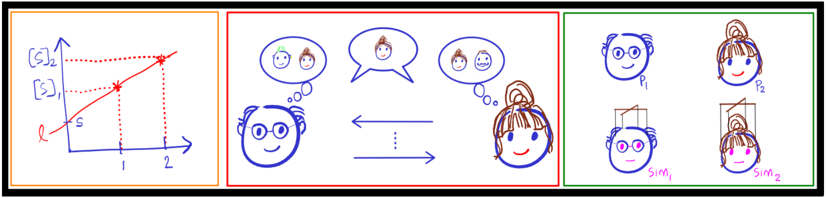

CS783: Theoretical Foundations of Cryptography

Lecture 17 (08/Oct/24)

Instructor: Chethan Kamath

Recall from Last Three Lectures

- Interactive proof
- Zero knowledge (ZK) proof
- ZK proof of knowledge

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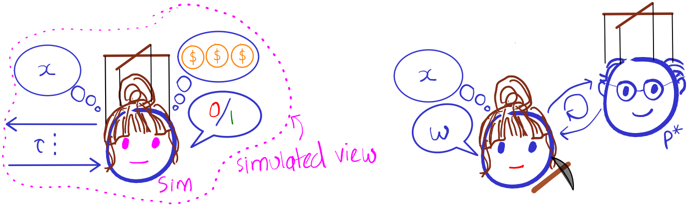
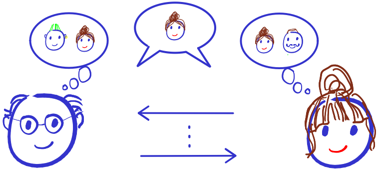


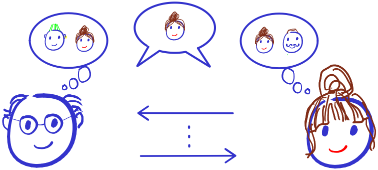
- Simulators and extractors
  - Key tools: out-of-order sampling and rewinding

- Main topic of Module III: private computation of functions



Plan for Today's Lecture...

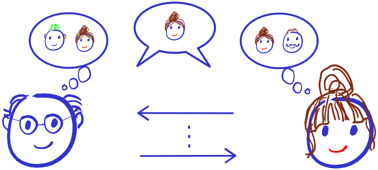
- Main topic of Module III: private computation of functions



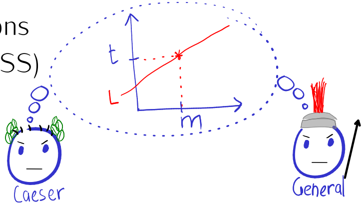
- Define syntax and security for the *two-party* case (2PC)
  - Extends the simulation paradigm

Plan for Today's Lecture...

- Main topic of Module III: private computation of functions



- Define syntax and security for the *two-party* case (2PC)
  - Extends the simulation paradigm
- Perfectly-private 2PC for *linear* functions
  - Key tool: threshold secret sharing (TSS)
  - Shamir's TSS



Plan for Today's Lecture...

General *template*:

1. Identify the task private computation of functions
2. Come up with precise threat model M (a.k.a security model)
   - Adversary/Attack: What are the adversary's capabilities?
   - Security Goal: What does it mean to be secure?
3. Construct a scheme Π
4. Formally prove that Π in secure in model M

General *template*:

private computation of functions

semi-honest model

1. Identify the task
2. Come up with precise threat model M (a.k.a security model)
   - Adversary/Attack: What are the adversary's capabilities?
   - Security Goal: What does it mean to be secure?

"static corruption"

"perfect privacy"

3. Construct a scheme Π
4. Formally prove that Π in secure in model M

General *template*:

1. Identify the task private computation of linear functions
2. Come up with precise threat model M (a.k.a security model) semi-honest model
   - Adversary/Attack: What are the adversary's capabilities? "static corruption"
   - Security Goal: What does it mean to be secure? "perfect privacy"
3. Construct a scheme Π secret-sharing-based
4. Formally prove that Π in secure in model M

   construct simulator

Private computation of functions is useful!

Apple's PSI

The Apple PSI System

Abhishek Bhowmick, Apple Inc.
Dan Boneh, Stanford University
Steve Myers, Apple Inc.
Kunal Talwar, Apple Inc.
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July 29, 2021

Mozilla's private aggregation

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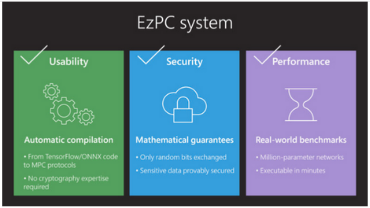
Microsoft's EzPZ



MP-SPDZ

data61 / **MP-SPDZ** Public

forked from bristolcrypto/SPDZ-2

Plan for Today's Lecture

1. Private Computation of Functions
2. New Tool: (Threshold) Secret Sharing
3. Computing Any Linear Function with Perfect Privacy

Plan for Today's Lecture

1 Private Computation of Functions

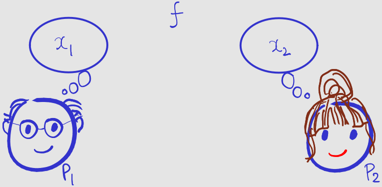
2 New Tool: (Threshold) Secret Sharing

3 Computing Any Linear Function with Perfect Privacy

Syntax of Two-Party Computation (2PC)

Defintion 1 (2PC protocol Π for $f : \mathcal{D}^2 \to \mathcal{R}$)

A ρ-round protocol $\Pi = (\mathsf{P}_1, \mathsf{P}_2)$ between two parties P_1 and P_2 with private input $x_1 \in \mathcal{D}$ and $x_2 \in \mathcal{D}$ and common output $y \in \mathcal{R}$



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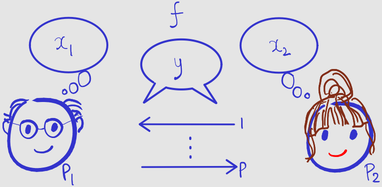

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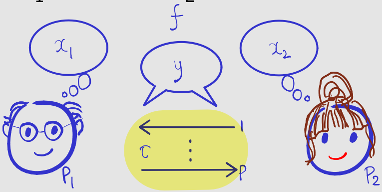

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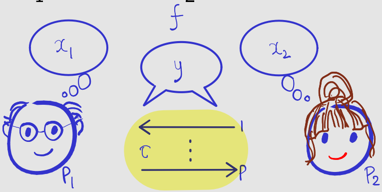

- *Correctness:* $\forall x_1, x_2 \in \mathcal{D}$, $\Pr[\langle P_1(x_1), P_2(x_2)\rangle \rightarrow f(x_1, x_2)] = 1$

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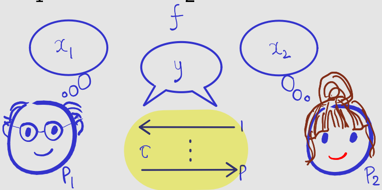

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- Can be more general:
  - Parties can have common input
  - Each party's output can be different
  - Two parties $\to$ n parties (MPC)

$P_2(x_2)$

$P_i(x_i)$

$(x_1)\ P_1$ (x)

$P_n(x_n)$

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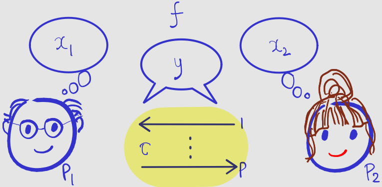

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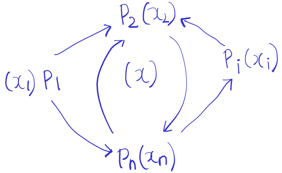

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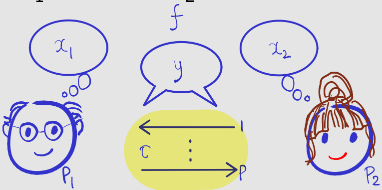

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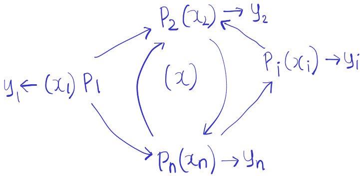

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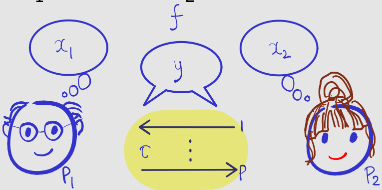

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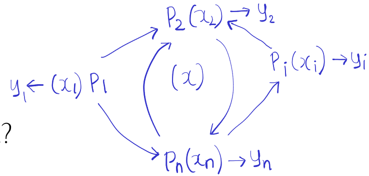
? How to frame ZKP as a 2PC protocol?



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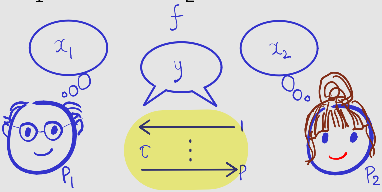

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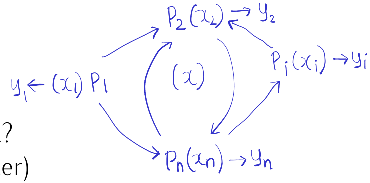
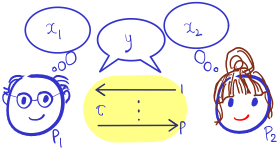


? How to frame ZKP as a 2PC protocol?
  - $P_1 :=$ P (prover) and $P_2 :=$ V (verifier)
  - $f := \mathcal{R}$, the NP relation $\Rightarrow$ common input$=x$, $x_1 = w$, $x_2 = \perp$

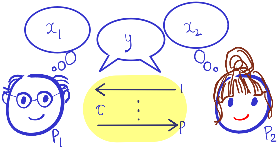
Let's Define Security...

? What are the requirements intuitively from P_1's perspective?



What are the requirements intuitively from P_1's perspective?

- P_2 should not learn anything about P_1's input x_1...



Let's Define Security...

What are the requirements intuitively from P_1's perspective?

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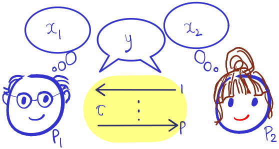

- How to formalise this?

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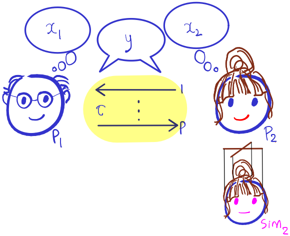
  - There exists a *simulator* for Sim_2's view



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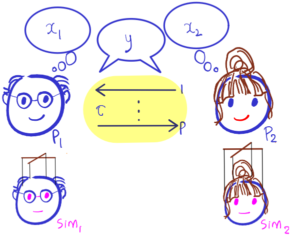

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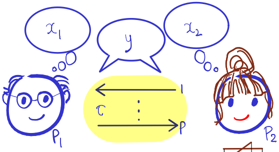

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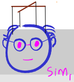

Defintion 2 (Semi-honest perfect security for 2PC)

Π *computes* f *with perfect privacy if there exists 1) a PPT simulator* Sim_2 *such that for all distinguishers* D *and for all* $x_1, x_2 \in \mathcal{D}$*, the following is zero*

$$\Pr[\mathsf{D}(\mathsf{View}_{\mathsf{P}_2}(\langle \mathsf{P}_1(x_1), \mathsf{P}_2(x_2)\rangle = 1] - \Pr[\mathsf{D}(\mathsf{Sim}_2)(x_2, y)) = 1]$$

and 2) a PPT simulator Sim_1 *such that....*

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- Extending the definition:
  - Semi-honest $\rightarrow$ malicious
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  - Semi-honest $\rightarrow$ malicious
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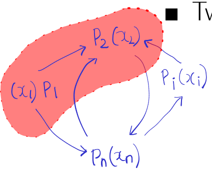
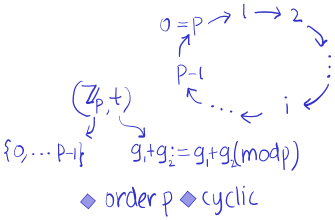
  - Two parties $\rightarrow$ n parties (MPC): t-privacy (for $t \leq n$)
    - Any fixed $(t-1)$-sized subset of parties $\mathcal{P}^* \subset [n]$ can be corrupt
    - There exists $\mathsf{Sim}_{\mathcal{P}^*}$ that simulates views of all parties in $\mathcal{P}^*$ given their inputs $\{x_i\}_{i \in \mathcal{P}^*}$ and the output y

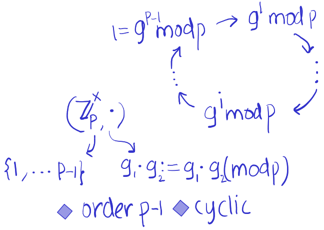


Plan for Today's Lecture

Addition modulo prime p

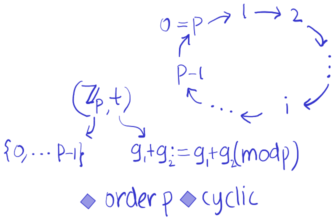


Multiplication modulo prime p

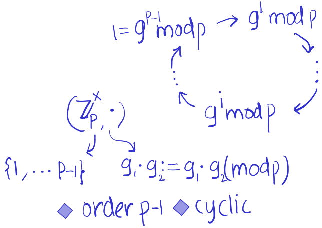


- Recall groups $(\mathbb{Z}_p, +)$ and $(\mathbb{Z}_p^\times, \cdot)$ from Lecture 08

Addition modulo prime p



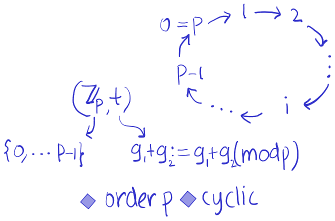
Multiplication modulo prime p



- Recall groups $(\mathbb{Z}_p, +)$ and $(\mathbb{Z}_p^\times, \cdot)$ from Lecture 08
  - $+$ and $\cdot$ are "compatible" with each other:
    - For any $a, b, c \in \mathbb{Z}_p$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$

Finite Fields

Addition modulo prime p



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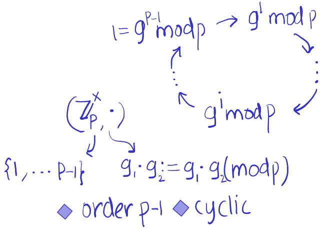

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  - $+$ and $\cdot$ are "compatible" with each other:
    - For any $a, b, c \in \mathbb{Z}_p$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$
- Can be combined to get a "field" $\mathbb{F}_p = (\mathbb{Z}_p, +, \cdot)$

Finite Fields

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0 = p$

$(\mathbb{Z}_p, +)$

$\{0, \dots p-1\}$ $\quad$ $g_1 + g_2 := g_1 + g_2 \pmod p$

◆ order p ◆ cyclic

Multiplication modulo prime p

$1 = g^{p-1} \bmod p \to g^1 \bmod p \to \cdots \to g^i \bmod p \to \cdots \to 1 = g^{p-1} \bmod p$

$(\mathbb{Z}_p^\times, \cdot)$

$\{1, \dots p-1\}$ $\quad$ $g_1 \cdot g_2 := g_1 \cdot g_2 \pmod p$

◆ order $p-1$ ◆ cyclic

- Recall groups $(\mathbb{Z}_p, +)$ and $(\mathbb{Z}_p^\times, \cdot)$ from Lecture 08
  - $+$ and $\cdot$ are "compatible" with each other:
    - For any $a, b, c \in \mathbb{Z}_p$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$
- Can be combined to get a "field" $\mathbb{F}_p = (\mathbb{Z}_p, +, \cdot)$
- Finite counterpart of real numbers $\mathbb{R}$
  - $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$ are groups
  - $+$ and $\cdot$ are distributive

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0$

$(\mathbb{Z}_p, +)$: $\{0, \dots, p-1\}$, $g_1 + g_2 := g_1 + g_2 \pmod p$

◆ order p ◆ cyclic

Multiplication modulo prime p

$1 = g^{p-1} \bmod p \to g^1 \bmod p \to \cdots \to g^i \bmod p \to \cdots \to 1$

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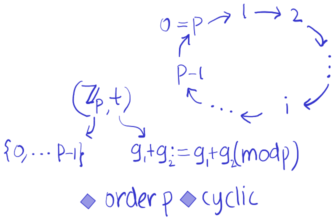
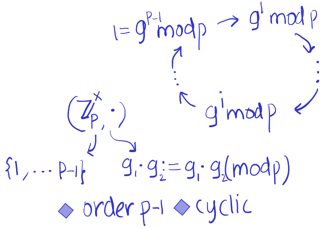
Addition modulo prime p



Multiplication modulo prime p



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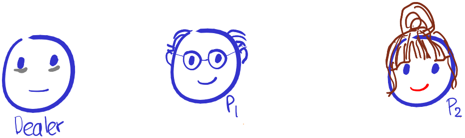
- $\mathbb{F}_2 = (\mathbb{Z}_2, +, \cdot)$ corresponds to Boolean algebra $(\{F, T\}, \oplus, \wedge)$

(t, n)-Threshold Secret Sharing (TSS)



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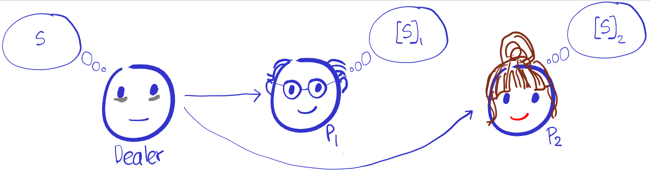

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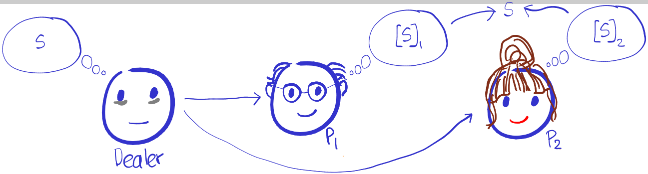

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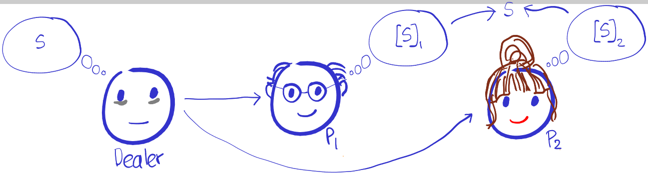

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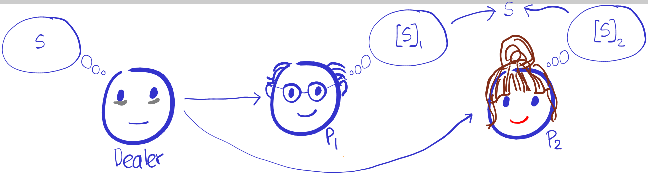

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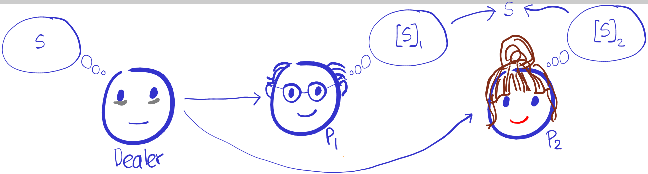

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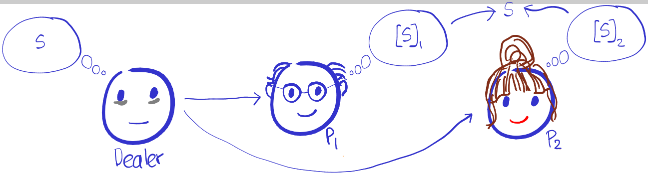

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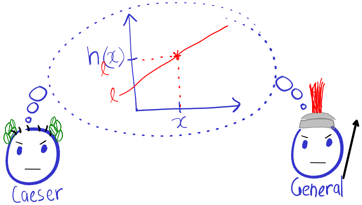
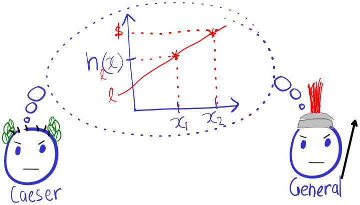
Exercise 1

What happens when you extend 3 to construct (t, n)-TSS for arbitrary n and $t \leq n$?



- Recall construction of pairwise-independent hash function $h_\ell(x) := \ell(x)$
  - Key is a "random line" ℓ over $\mathbb{F}_p$ and hash is evaluation on ℓ



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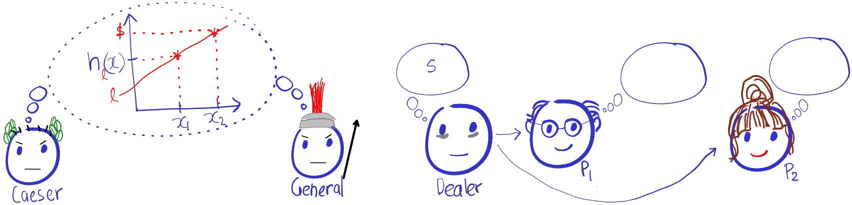

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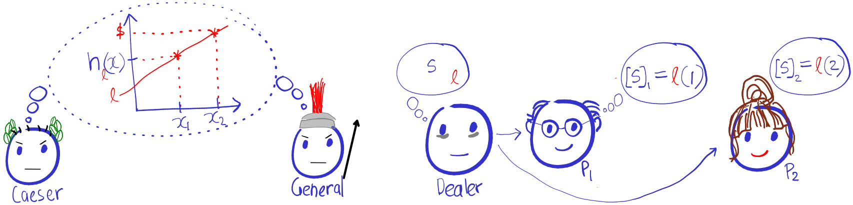
Shamir's (t, n)-TSS...



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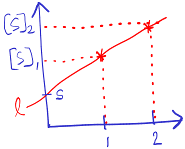

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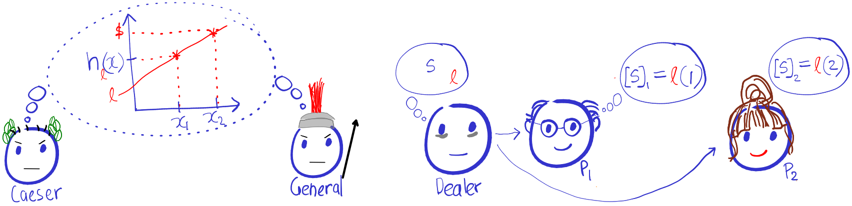

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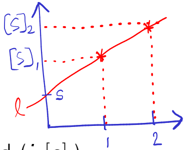
  1. Reconstruct ℓ by drawing line through $(i, [s]_i)$ and $(j, [s]_j)$
  2. Output $\ell(0)$



Shamir's (t, n)-TSS...

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+ The construction is "linear":
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  - $([s]_1, [s]_2)$ shares of $s \Rightarrow$ for $c \in \mathbb{F}_p$, $(c \cdot [s]_1 \bmod p, c \cdot [s]_2 \bmod p)$ shares of $s \cdot c \bmod p$

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- All of the above ideas extend to arbitrary n and $t \leq n$

Exercise 2 (Hint: use a degree-t polynomial instead of line)

Formally describe and prove Shamir's (t, n)-TSS

Plan for Today's Lecture

How to Privately Compute Linear Functions?...

Defintion 3

A function $\mathsf{f} : \mathbb{F}_p^n \to \mathbb{F}_p$ *is linear if* $\forall \bar{a}, \bar{b} \in \mathbb{F}_p^n : \mathsf{f}(\bar{a} + \bar{b}) = \mathsf{f}(\bar{a}) + \mathsf{f}(\bar{b})$

$\Rightarrow \forall c \in \mathbb{F}_p, \bar{a} \in \mathbb{F}_p^n : \mathsf{f}(c \cdot \bar{a}) = c \cdot \mathsf{f}(\bar{a})$

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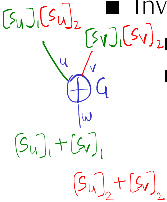
- Invariant: every party P_i will have secret share $[s_w]_i$ of wire w
  - To generate shares of output of $\oplus$, *add* shares of input wires
  - To generate shares of output of $\odot_c$, *multiply* share with c



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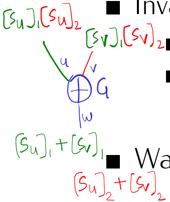

- Warm-up: let's privately compute $\oplus$ over $\mathbb{F}_2$

How to Privately Compute Linear Functions?...

Protocol 1 (Protocol Π for linear f : $\mathbb{F}_p^n \to \mathbb{F}_p$)

1. *Secret-share input:*
   1. *Each P_i chooses random degree-t polynomial q_i with $q_i(0) = x_i$*
   2. *Each P_i sends share $[x_i]_j := q_i(j)$ to P_j (for all $j \neq i$)*

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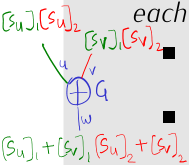

   - *If $G_k = \oplus$: define share of G_k's output wire to be sum of P_i's shares of G_k's input wires*
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   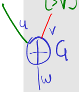

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   $[s_u]_1 + [s_v]_1, [s_u]_2 + [s_v]_2$
3. *Reconstruct output: each party P_i*
   1. *Broadcasts its share of output wire to parties P_j, $j \neq i$*
   2. *Reconstructs q_i from all shares of output wire*
   3. *Outputs $q_i(0)$*

Π Computes f with Perfect Secrecy

Theorem 1

Assuming $(\boldsymbol{t}, \boldsymbol{n})$-linear TSS, Π computes f *with $\boldsymbol{t}$-privacy*

Proof (Sketch).

- Idea: $< \boldsymbol{t}$ parties $\mathcal{P}^*$ corrupt $\Rightarrow$ inputs of $[\boldsymbol{n}] \setminus \mathcal{P}^*$ perfectly hidden by security of TSS

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- Simulator $\mathsf{Sim}(\mathcal{P}^*, \{x_i\}_{i \in \mathcal{P}^*}, y)$
  - Share input:
    1. For every $i \in \mathcal{P}^*$, sample random q_i with $q_i(0) = x_i$ (as in Π)
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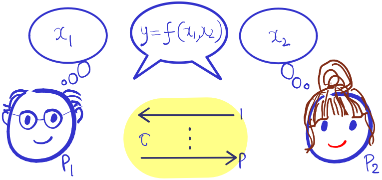
  - Emulate circuit: for each non-output gate G_k in top. order
    - Generate shares of G_k's output wire as in Π
  - Program output:
    - Set polynomial q_o of output gate consistently with its input wires and with $q_o(0) = y$ □

To Recap Today's Lecture

- Task 6: Private computation of functions



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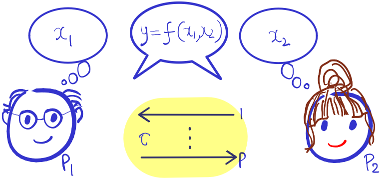

- Defined syntax and security for the *two-party* case (2PC)
  - Extends the simulation paradigm

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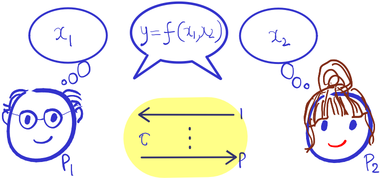
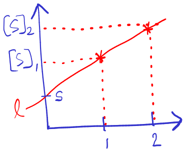


- Defined syntax and security for the *two-party* case (2PC)
  - Extends the simulation paradigm
- Perfectly-private MPC for *linear* functions
  - Key tool: threshold secret sharing (TSS)
  - Shamir's TSS
  - Key idea: "compute over shares"



Next Lecture

- Continue with Task 6

  ⚠ Perfectly-private 2PC for *general* functions is impossible!

  - Counter-example: $\wedge$

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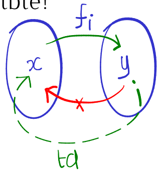
  - What do we do? Relax to computational privacy
  - New tool: oblivious transfer
    - Oblivious transfer from trapdoor permutations



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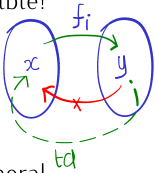
  ⚠ Perfectly-private 2PC for *general* functions is impossible!
    - Counter-example: $\wedge$
  - What do we do? Relax to computational privacy
  - New tool: oblivious transfer
    - Oblivious transfer from trapdoor permutations
  - GMW protocol: computationally-private MPC for general functions



References

1. MPC was first studied in [GMW87], building on [GMR89]
2. Shamir's TSS is from [Sha79]
3. The perfectly-secure MPC protocol for linear functions described here is taken from [AL17, §4.2]

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