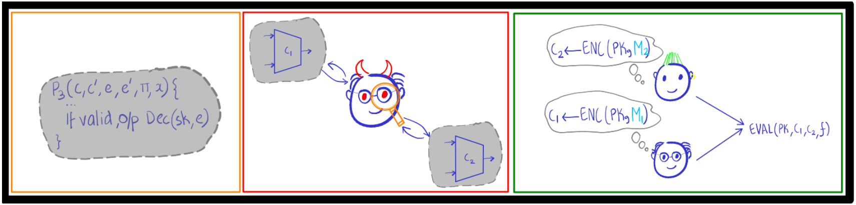

CS783: Theoretical Foundations of Cryptography

Lecture 24 (08/Nov/24)

Instructor: Chethan Kamath

Recall from Last Two Lectures...

- Program obfuscation: "scramble/encrypt" a program such that
  1. functionality preserved
  2. hard to "reverse engineer"

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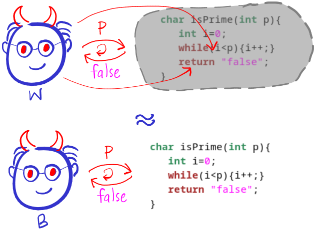
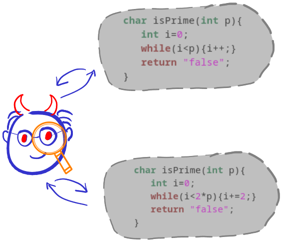
  - Lecture 23: Indistinguishability obfuscation (IO)





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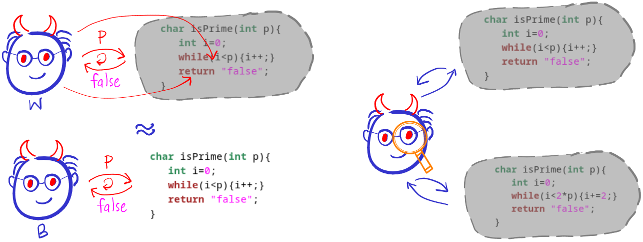
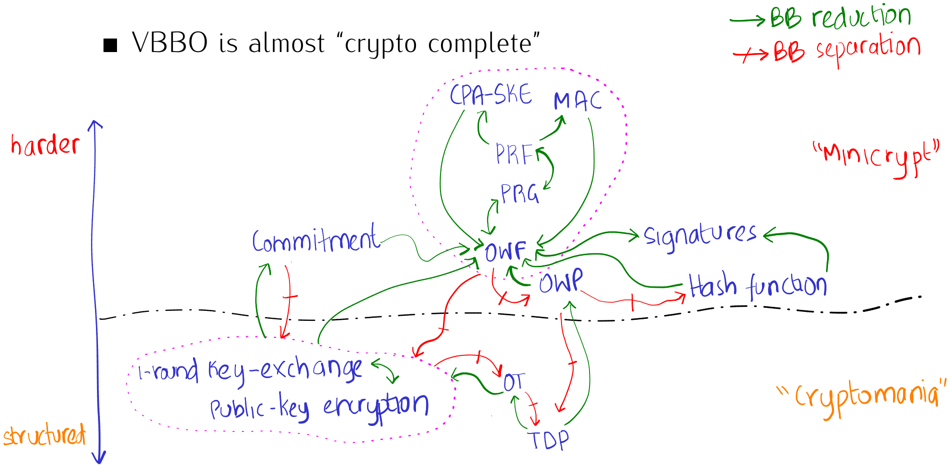


+ Bypassed black-box separations exploiting primitive's program
  - OWF $\xrightarrow{\text{VBBO}}$ OWP and PRG $\xrightarrow{\text{IO}}$ PKE

Recall from Last Two Lectures...

- VBBO is almost "crypto complete"



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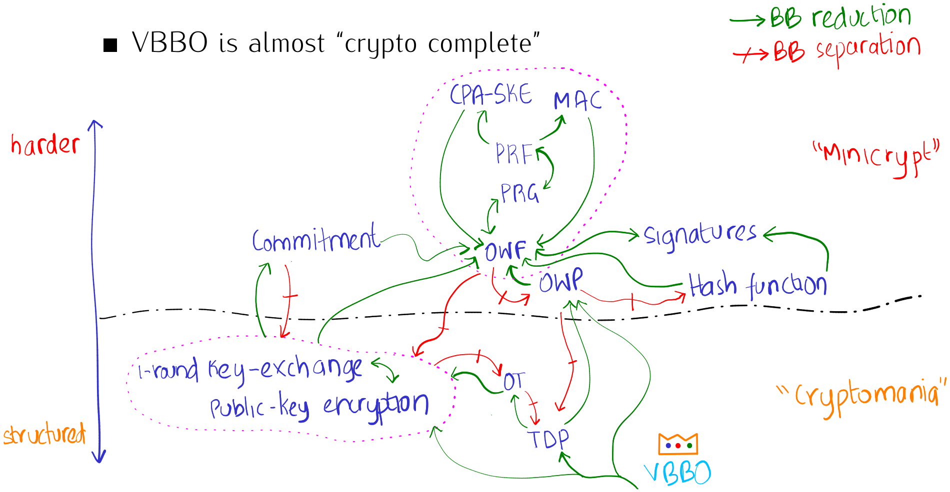

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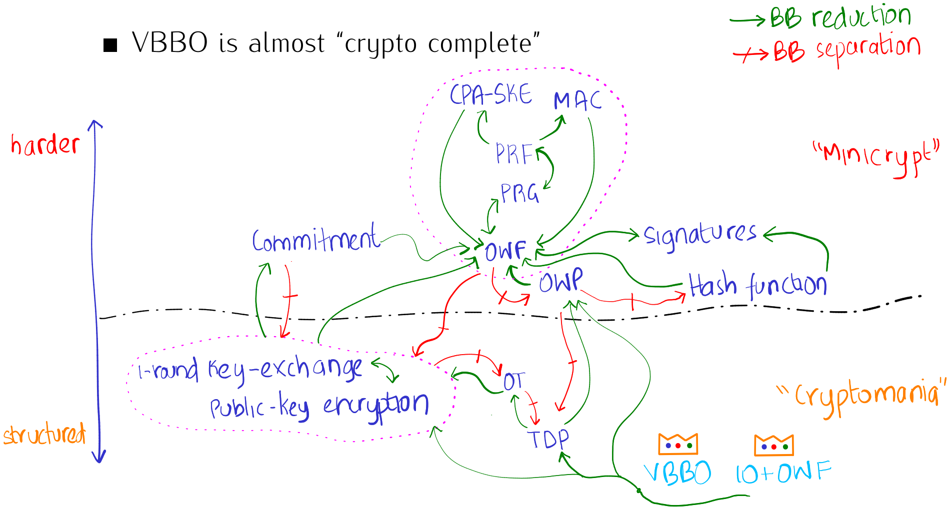

- IO + OWF also yields most of crypto!

Plan for Today's Lecture...

- VBBO for general programs is impossible

$$P^*_{\alpha,\beta,\gamma}(b,x) := \begin{cases} \Delta_{\alpha,\beta}(x) & \text{If } b=0 \\ S_{\alpha,\beta,\gamma}(x) & \text{If } b=1 \end{cases}$$

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  - Boosting theorem for IO: fully homomorphic encryption (FHE) + IO for "shallow" circuits $\rightarrow$ IO for all circuits

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- What about IO for general programs?
  - Boosting theorem for IO: fully homomorphic encryption (FHE) + IO for "shallow" circuits $\rightarrow$ IO for all circuits
  - State of affairs for IO for "shallow" circuits

Plan for Today's Lecture...

1 Boosting IO Using FHE

2 Constructing IO for NC^1: What Do We Know?

Recall... (Fully) Homomorphic Encryption (Lecture 19)

- Public-key encryption + public *evaluation* algorithm

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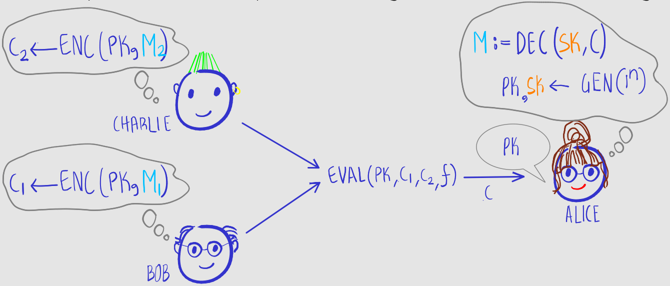
Defintion 1 (Homomorphic encryption (HE) for function class $\mathcal{F}$)

A PKE $\Sigma = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec}) + \mathsf{Eval}$ *algorithm with following syntax*



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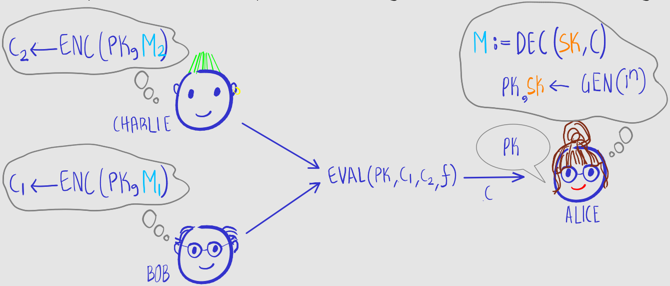

- *Compactness of evaluation:* $|c|$ *obtained from* Eval *independent of* $|f|$
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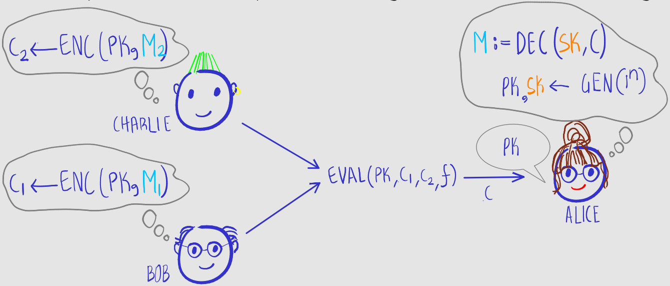

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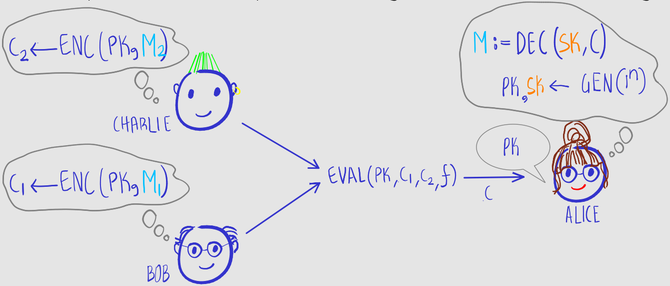

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- *Fully* HE: $\mathcal{F}$=functions computable by poly.-sized circuits
- GSW construction: FHE that is secure assuming LWE

Recall… IO for Circuits (Lecture 23)

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Defintion 2 (Indistinguishability obfuscator (IO) for circuit class $\mathcal{C}$)

A PPT algorithm Obf *that takes as input any circuit* $\mathsf{C} \in \mathcal{C}$ *and security parameter* n, *and outputs obfuscated circuit* C *such that:*

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1. *Functionality preserved*
2. *Slowdown is polynomial*
3. *IO security: for every functionally-equivalent, same-sized* $\mathsf{C}_1, \mathsf{C}_2 \in \mathcal{C}$ *and PPT* D, *the following is negligible:*

$$\left| \Pr_{\tilde{C}_1 \leftarrow \mathsf{Obf}(1^n, C_1)}\left[D(\tilde{C}_1) = 1\right] - \Pr_{\tilde{C}_2 \leftarrow \mathsf{Obf}(1^n, C_2)}\left[D(\tilde{C}_2) = 1\right] \right|$$

Boosting IO for NC^1 using FHE...

- Goal: Obf for NC^1 + FHE $\Pi \rightarrow \mathsf{Obf}'$ for all circuits
- High-level idea: use FHE to encrypt *circuit* and then use Obf to "decrypt-then-evaluate"

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  - $\mathsf{Obf}'(\mathsf{C})$ consists of the following:
    1. FHE ciphertext c of C under pk
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$P_1(c, x)\{$
  $C := Dec(sk, c)$ hardwired
  Output $C(x)$
$\}$

Boosting IO for NC^1 using FHE...

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  - To evaluate (c, Obf(P_1)) on x, output Obf(P_1)(C, x)

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$P_1(c, x)\{$

$\quad C := \mathrm{Dec}(sk, c)$ → hardwired

$\quad$ Output $C(x)$

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Problem: ?

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High-level idea: use FHE to encrypt *circuit* and then use **Obf** to "decrypt-then-evaluate"

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- Attempt 1:
  - **Obf$'$**(C) consists of the following:
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```
P1(c, x) {
    C := Dec(sk, c)      hardwired (sk)
    Output C(x)
}
```

  - To evaluate (c, **Obf**(P_1)) on x, output **Obf**(P_1)(C, x)

⚠ Problem: **Obf** does not support evaluation of C

not necessarily in NC^1

Boosting IO using FHE...

- Attempt 2: let's exploit homomorphic evaluation
  - Obf′(C) consists of the following:
    1. FHE ciphertext c of C under pk
    2. Obf(P_2), where P_2 is the following decrypt-and-output function

$P_2(e)\{$

$\quad y := \mathrm{Dec}(\overset{\text{hardwired}}{sk}, e)$

$\quad$ Output y

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  - To evaluate (c, Obf(P_2)) on x, homomorphically evaluate

$$e := \mathsf{Eval}(pk, c, \mathsf{U}(\cdot, x))$$

universal circuit
∀circuit C, input x:
$U(C,x) = C(x)$

and then output Obf(P_2)(e)

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universal circuit
$\forall$ circuit C, input x:
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⚠ Problem: insecure as P_2 decrypts all ciphertexts

Boosting IO using FHE...

- Solution: only decrypt certain "constrained" ciphertexts

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  1. Two FHE ciphertexts (c, c') of C, under pk and pk'
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$P_3(c, c', e, e', \pi, x)\{$

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Verify computation of e, e′ using π

If valid, o/p Dec(sk, e)

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hardwired

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- To evaluate $\mathsf{Obf}'(\mathsf{C}) := (c, c', \mathsf{Obf}(\mathsf{P_3}))$ on input x
  1. Evaluate $e := \mathsf{Eval}(pk, c, \mathsf{U}(\cdot, x))$ and $e' := \mathsf{Eval}(pk', c', \mathsf{U}(\cdot, x))$
  2. Let π be wire values during computation of e and e'
  3. Output $\mathsf{Obf}(\mathsf{P_3})(c, c', e, e', \pi, x)$

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Exercise 1

Show that verifying π can be carried out in NC^1

Boosting IO using FHE...

Theorem 1

If Obf *is IO for* NC^1 *and* Π *is an FHE then* Obf' *is IO for all circuits*

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Proof Sketch (Hybrid argument).

Left world H_0 — Goal — Right world H_4

$(c_0, c_0'), \; P_3(c, c', e, e', \pi, x)\{ \; \ldots \;$ If valid, o/p $\mathrm{Dec}(sk, e) \; \}$ $\approx$ $(c_1, c_1'), \; P_3(c, c', e, e', \pi, x)\{ \; \ldots \;$ If valid, o/p $\mathrm{Dec}(sk, e) \; \}$

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□

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$\approx$

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Hybrid H_2

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$\approx$ Goal

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$\approx_{FHE}$

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$\approx_{IO}$

Hybrid H_2

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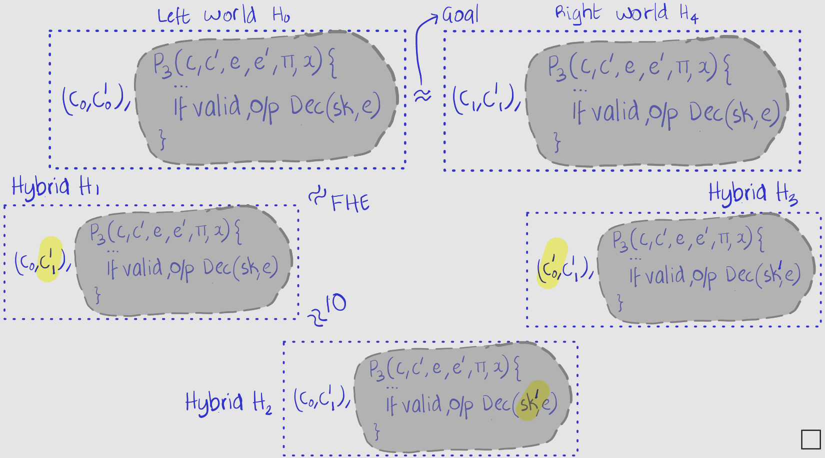

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$\approx$ Goal

Right world H_4

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FHE $\approx$

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Right world H_4

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$\approx$ FHE

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$\approx$ IO

Hybrid H_2

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FHE $\approx$

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IO $\approx$

$\square$

Digression: Bootstrapping FHE

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- $\mathsf{Gen}'(1^n)$*:*
  - *Generate* $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$ *and compute* $c_{sk} := \mathsf{Enc}(pk, sk)$
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- Enc' *and* Dec' *are same as* Enc *and* Dec*, respectively*

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Exercise 2

Show that Π' *is FHE for all circuits if* Π *is "circular secure" FHE for* NC^1

Plan for Today's Lecture

1. Boosting IO Using FHE

2. Constructing IO for NC^1: What Do We Know?

- Bilinear map:

$$\mathrm{e} : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_T$$

such that for every $g_1, g_2 \in \mathbb{G}$ and $a, b \in \mathbb{Z}_p$,

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⚠ Problem: we don't know how to construct even trilinear maps

- All proposals of multilinear maps were later broken

LWE + "Local" PRG + Bilinear Maps $\rightarrow$ IO for NC^1

+ Recent result relaxes the assumptions to
  1. Learning with errors (LWE)
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  3. "Local" PRG: each output bit of the PRG only depends only on a "few" input bits

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- Open:
  - LWE $\rightarrow$IO for NC^1
  - Simpler constructions from stronger assumptions

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- We started with black-box separations: OWF$\not\rightarrow$OWP
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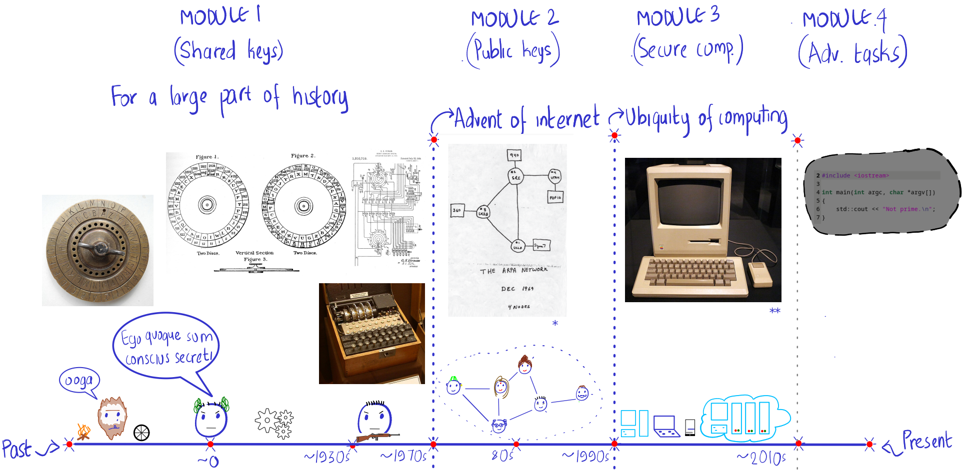
```
C(x){
   /*Hardwired: pk, m */
   if(G(x)=pk) output m
   else output "⊥"
}
```

IO for NC^1 $\xrightarrow{\text{FHE}}$ IO for P/poly

FHE for NC^1 $\longrightarrow$ FHE for P/poly

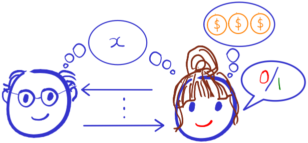
- Key tools/ideas: how to hide secrets using IO, punctured programming, bootstrapping/boosting
- Takeaways: separations are useful (they pin point our limits)

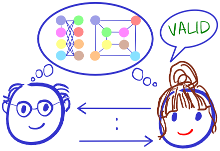
To Recap Module IV...



Spring'25: Introduction to Probabilistic Proof Systems

- Module III: Interactive proofs (IP), zero-knowledge proofs (ZKP)





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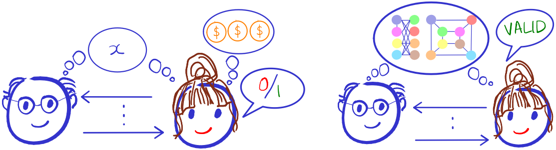

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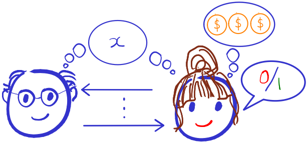

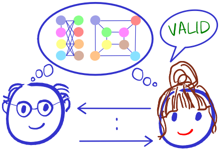

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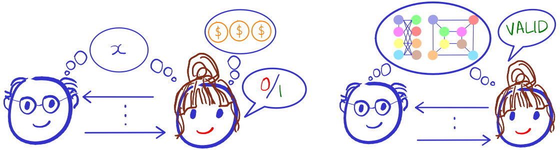

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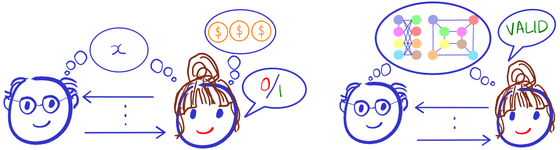

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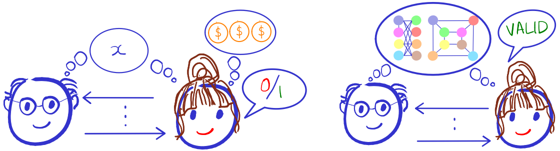

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- Will send a link to course website via Moodle

References

1. The boosting result for IO is from [GGH+13]. The presentation here is from Lecture 13 of Mark Zhandry's COS597C course (Fall 16).
2. The bootstrapping result for FHE is due to Gentry [Gen09]
3. The construction of IO from multilinear maps can be found in [GGH+13]; the second construction can be found in [JLS21].

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