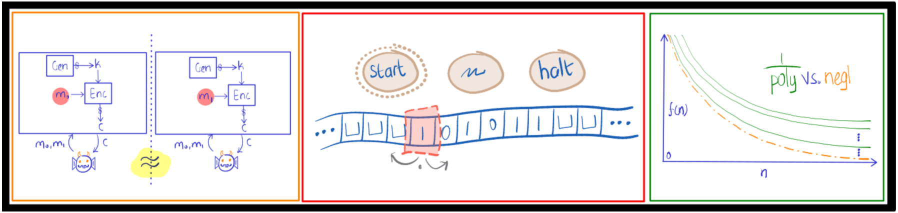

CS409m: Introduction to Cryptography

Lecture 04 (08/Aug/25)

Instructor: Chethan Kamath

Annonuncements

- Lab Exercise 1 (graded) will be out today (08/Aug)
  - Will be discussed in TA session today
- You should have registered on https://cs409m.ctfd.io/
  - See Moodle announcement by Nilabha for instructions

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- No lectures next week :(/ :)
  - Open house on 13/Aug and Ind. Day on 15/Aug

Recall from Previous Lecture...

- Task: secure communication with shared keys
- Threat model: perfect secrecy against eavesdroppers

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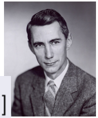
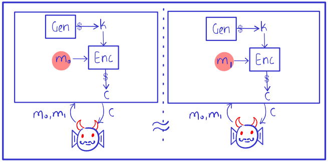
Break Model: Perfect secrecy

- Eve learns *no information* about the message
  1. Shannon's definition



$$\Pr[\mathrm{M} = m^* | \mathrm{C} = c^*] = \Pr[\mathrm{M} = m^*]$$

  2. Two worlds definition



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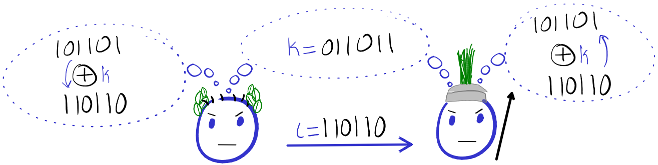

+ Theorem 1 (Shannon): one-time pad (OTP) is perfectly secret

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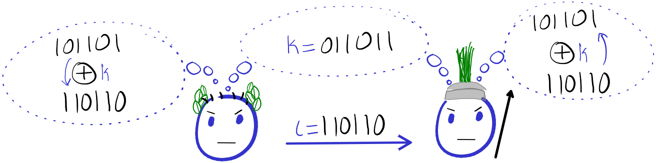

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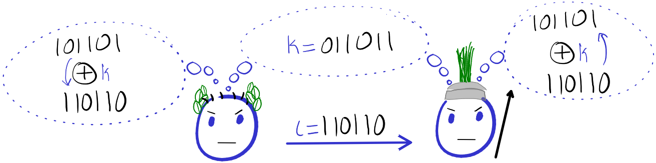

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− Theorem 2 (Shannon): For *any* perfectly-secret SKE, $|\mathcal{K}| \geq |\mathcal{M}|$

Plan for Today's Lecture...

- Bypass Shannon's barrier by *relaxing* the threat model:
  - "Impossible to break" $\rightarrow$ "Infeasible to break"

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- Eve may learn information, but only with "low probability" NEW
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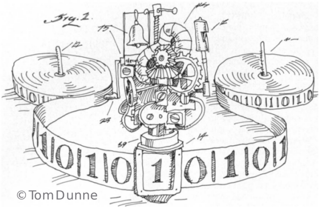
Both relaxations are necessary!

Plan for Today's Lecture...

Models of Computation



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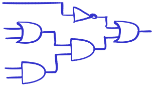

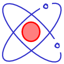

Negligible Functions

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Plan for Today's Lecture...

1. Models of Computation: A Primer
2. Negligible Functions
3. Computational Secrecy Against Eavesdroppers

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  - How to define running time?
  - What does "efficient" mean?
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  - ...
- E.g.: Turing machine (TM)
  - Introduced by Turing as "automatic machine"

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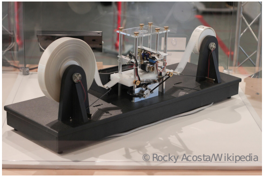

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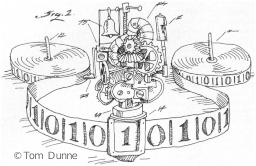

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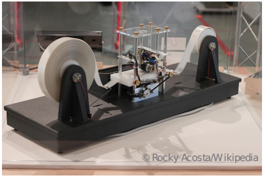

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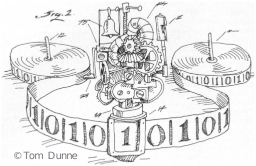
- E.g.: Turing machine (TM)
  - Introduced by Turing as "automatic machine"
  - Mathematically precise model of computation
- Components:
  - Tapes: to provide input, for memory...
  - States: "halt" "good so far"
  - Transition function/rule: "processor"



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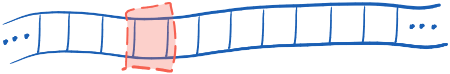
Definition 1 ([AB09], §1.2)

A k-tape Turing Machine M is described by a tuple (Γ, Q, τ) such that:

How Turing Machines Work...



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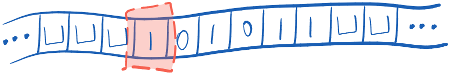

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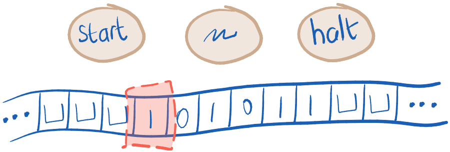

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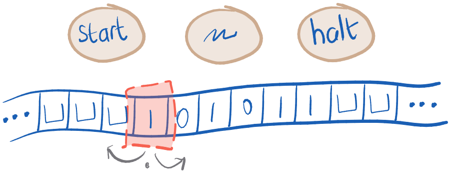

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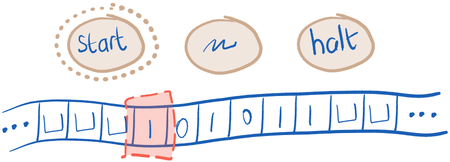
- τ is a function from $Q \times \Gamma^k$ to $Q \times \Gamma^k \times \{\rightarrow, \leftarrow, \cdot\}^k$
  - Transition function/rule: encodes behaviour of M



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  - The tape is initialised with the input string
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  - The head is at the start of the input
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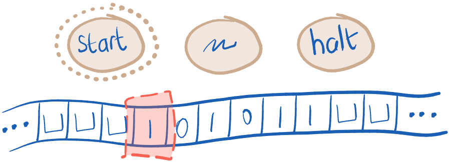

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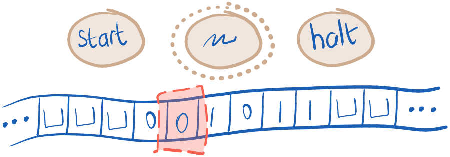
- Computation step
  - Apply τ on current state and input to obtain next state, output and next head position



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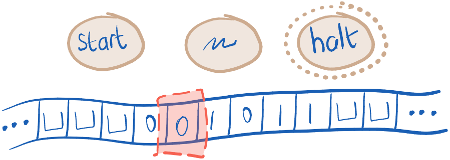

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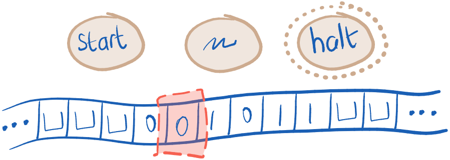

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Demo: turingmachine.io

Running Time of Turing Machine

Definition 2

Let $f : \{0,1\}^* \to \{0,1\}^*$ and $T : \mathbb{N} \to \mathbb{N}$ be functions, and M be a TM

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What was the running time of the TM in the demo?

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  - E.g., $T(n) := n^3$ or $T(n) = 4n^{1000} + \log(n)$

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  - E.g., $T(n) := n^3$ or $T(n) = 4n^{1000} + \log(n)$
- Efficient *randomised* computation $$$
  - Also referred to *probabilistic polynomial time* (PPT) ☆
  - Definition 2 extended to *randomised* TM

Other Models of Computation Exist

? Is your laptop a Turing Machine?

Other Models of Computation Exist

Is your laptop a Turing Machine? Not quite, closer to

- RAM Machine
  - TM with fixed-sized tape
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What about a basic calculator?

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- RAM Machine
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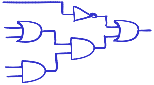
What about a basic calculator? Closer to

- Boolean circuit (family)
  - Represented using gates (AND, OR, NOT) and wires
  - One circuit for each input length
  - Size of the circuit is the number of its gates
  - Efficient circuits: size is polynomial (in input length)



Compromise I: We Restrict Eve to PPT TM

- Why TM? Church-Turing thesis:
  - *"Every physically realizable computation device – whether it's based on silicon, DNA, neurons, or some other alien technology – can be simulated (efficiently) by a Turing machine."** ([AB09])

*Possible exceptions: Boolean ciruit family, quantum TM

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  - Real-world adversaries assumed to be efficient
  - Polynomials have nice closure properties
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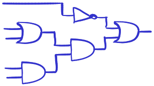
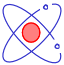


- Some stronger models for Eve:
  - Polynomial-sized family of circuits: allows "non-uniform" advice
  - Quantum polynomial-time algorithms



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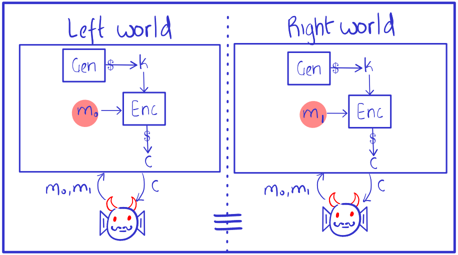
First Attempt at Computational Secrecy

Definition 3 (Recall: Two-Worlds Definition)

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is perfectly-secret if *for every* eavesdropper *Eve* and every message-pair $(m_0, m_1) \in \mathcal{M}$:

$$\Pr_{\substack{k \leftarrow \mathsf{Gen} \\ c \leftarrow \mathsf{Enc}(k, m_0)}}[\mathsf{Eve}(c) = 0] - \Pr_{\substack{k \leftarrow \mathsf{Gen} \\ c \leftarrow \mathsf{Enc}(k, m_1)}}[\mathsf{Eve}(c) = 0] = 0$$



First Attempt at Computational Secrecy

Candidate Defintion 1 (Computational Secrecy)

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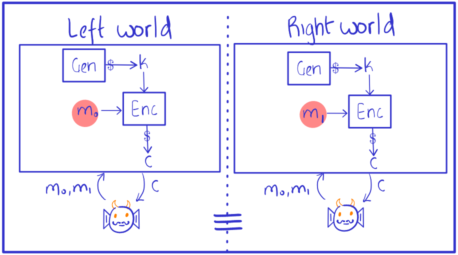

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Exercise 1

Show that Shannon's impossibility extends to Candidate Defintion 1

- Hint 1: use similar approach as in proof of Theorem 2 (Lecture 03)
- Hint 2: exploit randomness for efficiency

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Take-away: even Eve can distinguish with "very low" probability

Models of Computation

Negligible Functions

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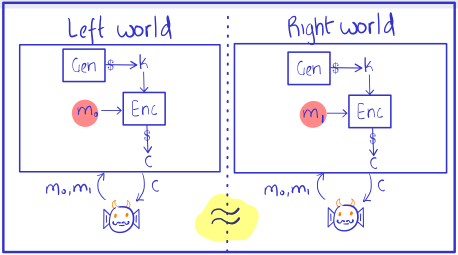

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Exercise 2

- Does Shannon's impossibility extend also to Candidate Defintion 2?
  - Hint: Work out precise probability of learning in Exercise 1

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- Does Shannon's impossibility extend also to Candidate Defintion 2?
  - Hint: Work out precise probability of learning in Exercise 1
- Can Eve trivally succeed with $1/|\mathcal{K}|$ probability? (Hint: guess the key?)

Take-away: $1/|\mathcal{K}|$ too low

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  - Intuitive def. of negligible function: function eventually smaller than *every* inverse polynomial

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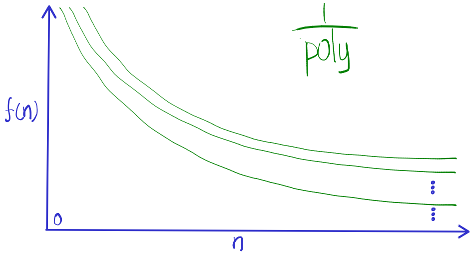

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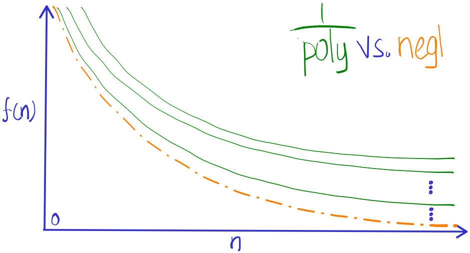

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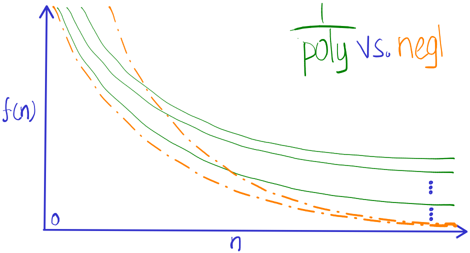

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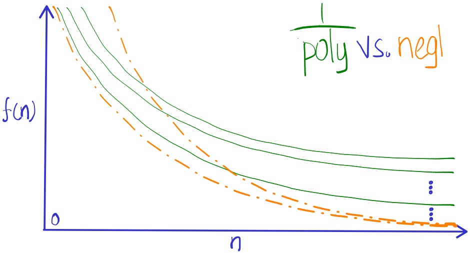

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+ Plus, like PPT have nice closure properties

Second Compromise: Break is a Low-Probability Event

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To show that $f(n)$ is *non-negligible*, show that there exists a polynomial p such that $f(n) > 1/p(n)$ for *infinitely often* ns.

Plan for Today's Lecture

1. Models of Computation: A Primer
2. Negligible Functions
3. Computational Secrecy Against Eavesdroppers

The Security Parameter

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  - Restrict to PPT Eve
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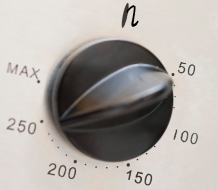

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  - Want: Honest algorithms run in time *fixed polynomial* in n
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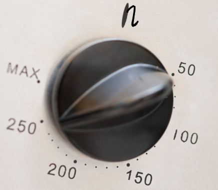

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- n is the "security parameter"
  - Determines amount of time (generally resources) required to "break" scheme

How to Choose the Right Security Parameter?...

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- Suppose a cryptography designer claims that Eve running in n^3mins can break his scheme with probability $2^{40}/2^n$

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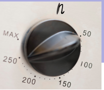

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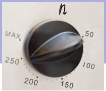

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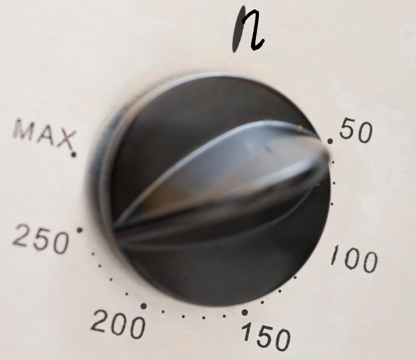

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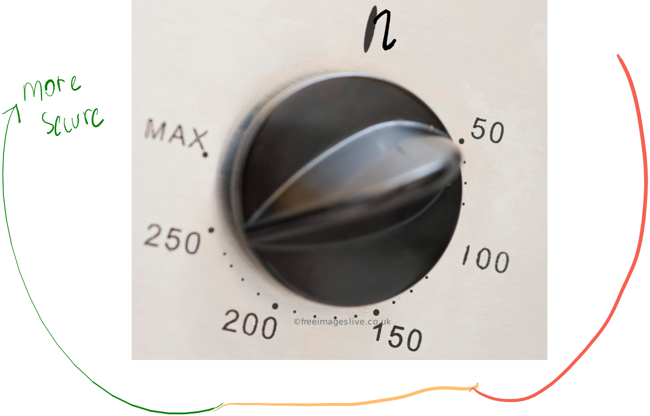

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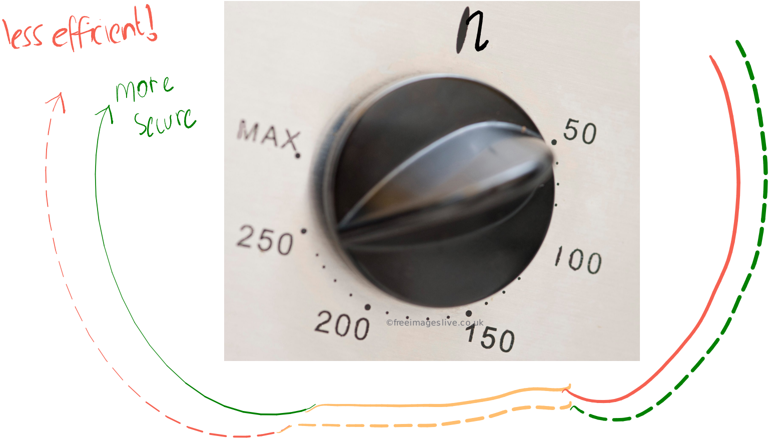
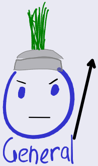


Incorporating Security Parameter into SKE Definition

Definition 4 (Shared/Symmetric-Key Encryption (SKE))

An SKE Π is a triple of efficient algorithms $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ with the following syntax:



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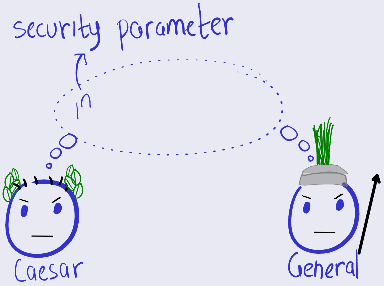

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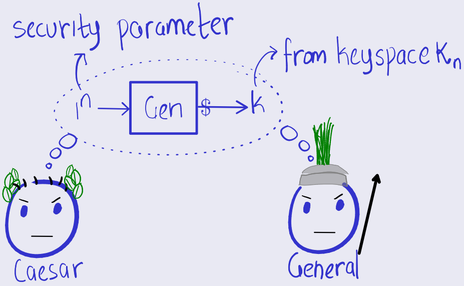

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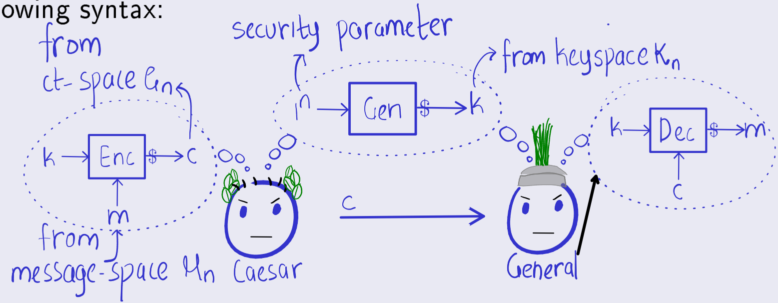

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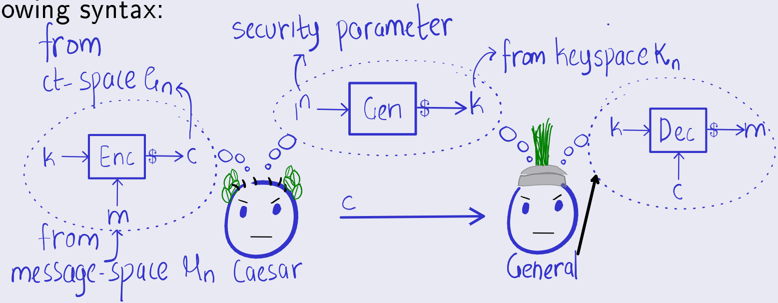

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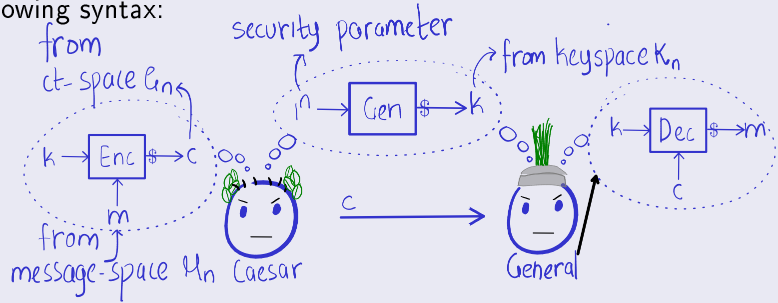
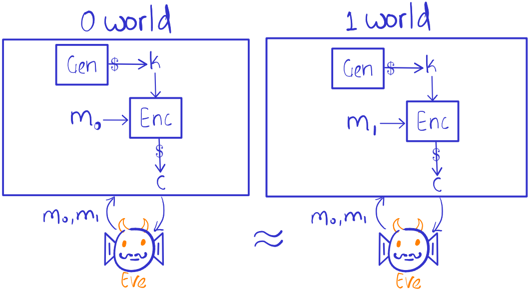


- Correctness of decryption: for every $n \in \mathbb{N}$, message $m \in \mathcal{M}_n$,

$$\Pr_{k \leftarrow \mathsf{Gen}(1^n), c \leftarrow \mathsf{Enc}(k,m)}[\mathsf{Dec}(k, c) = m] = 1$$

Let's Finally Define Computational Secrecy!



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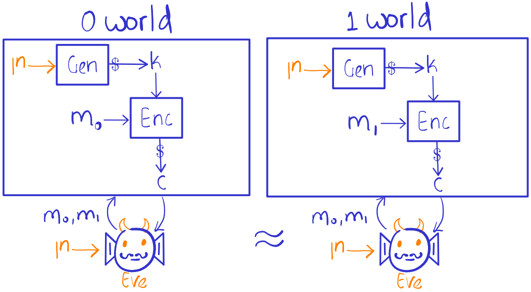

Definition 5 (Two-Worlds Definition)

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is computationally-secret against eavesdroppers if for *every* PPT Eve

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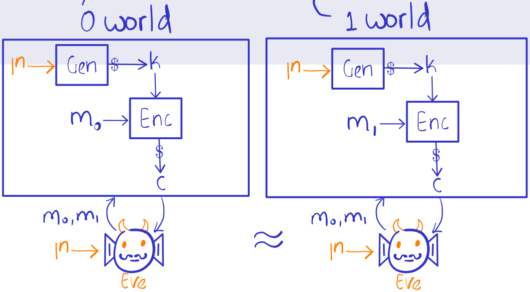

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Exercise 3

Does Definition 5 change if we quantify for all pair of messages (m_0, m_1) instead of adversarially choosing it?

More Generally: Computational Indistinguishability

Definition 6 (computational indistinguishability)

Two distributions X_0 and X_1 are *computationally indistinguishable* if for every PPT *distinguisher* D,

$$\delta(n) := \Pr_{x \leftarrow \mathrm{X}_0}[\mathsf{D}(x) = 0] - \Pr_{x \leftarrow \mathrm{X}_1}[\mathsf{D}(x) = 0]$$

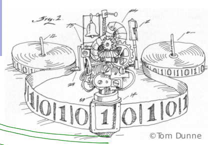
is negligible.

- Computational secrecy against eavesdroppers can be rephrased as: the ciphertext distribution in the left and the right worlds are computationally indistinguishable.

Exercise 4

Formally show the above

Recap/Next Lecture



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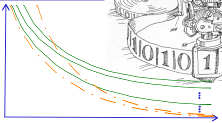

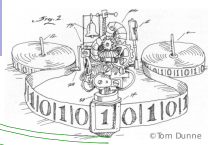

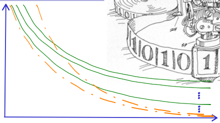

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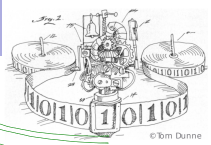

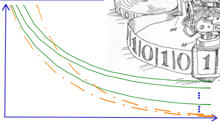

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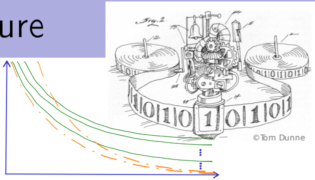

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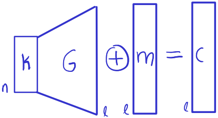

- Next lecture:
  - Pseudorandom generators (PRG)
  - Computationally-secret SKE scheme: "Computational OTP"
  - First security reduction!

More Questions?

References

1. §3.1 in [KL14] for more details on computational secrecy
2. Chapter 1 in [AB09] for more about Turing machines. The original paper is [Tur37]
3. turingmachine.io for visualisation of Turing machines

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