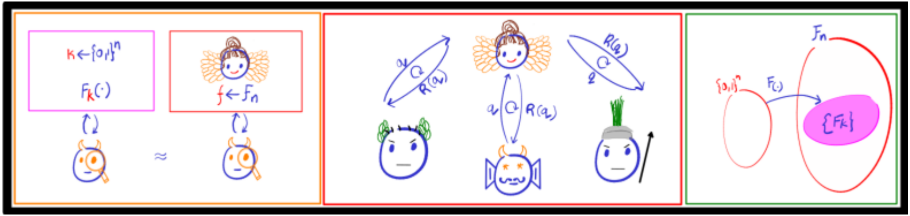


CS409m: Introduction to Cryptography

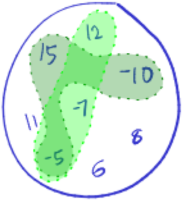
Lecture 07 (22/Aug/25)

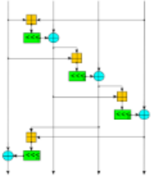
Instructor: Chethan Kamath

Recall from Previous Lecture

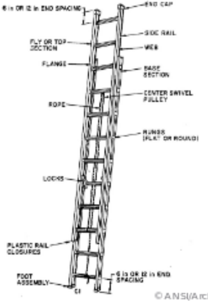
- Task: secure communication of very *long messages* with shared keys
- Threat model: computational secrecy against eavesdroppers

How to Construct PRGs?





PRG Length Extension



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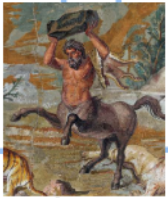
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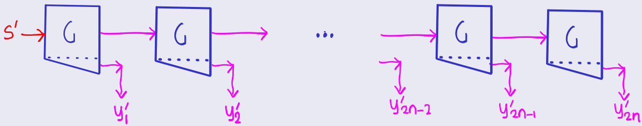
Main tool: hybrid argument

Recall from Previous Lecture...

Theorem 1

If G is a PRG, then so is G'.

Proof. $\exists$ distinguisher D for G $\Leftarrow$ $\exists$ distinguisher D' for G'.



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Recall from Previous Lecture...

Theorem 1

If G is a PRG, then so is G′.

Proof. ∃ distinguisher D for G ⇐ ∃ distinguisher D′ for G′.


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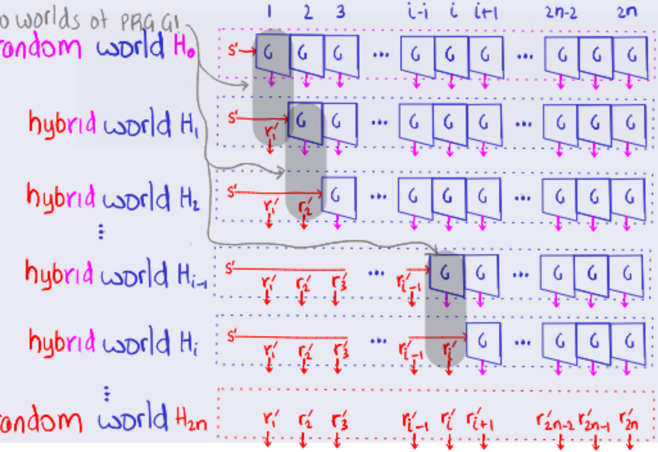

Recall from Previous Lecture...

Theorem 1

If G is a PRG, then so is G′.

Proof. ∃ distinguisher D for G ⇐ ∃ distinguisher D′ for G′.

Two worlds of PRG G!

pseudorandom world H_0

hybrid world H_1

hybrid world H_2

⋮

hybrid world H_{i-1}

hybrid world H_i

⋮

random world H_{2n}

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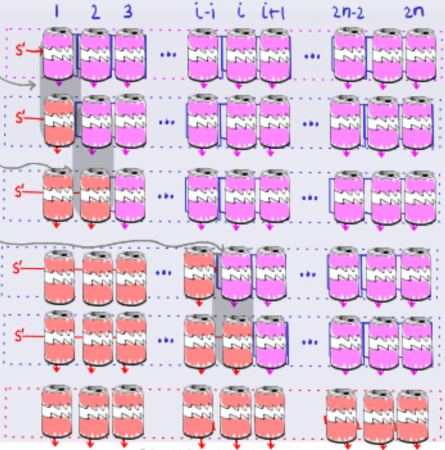

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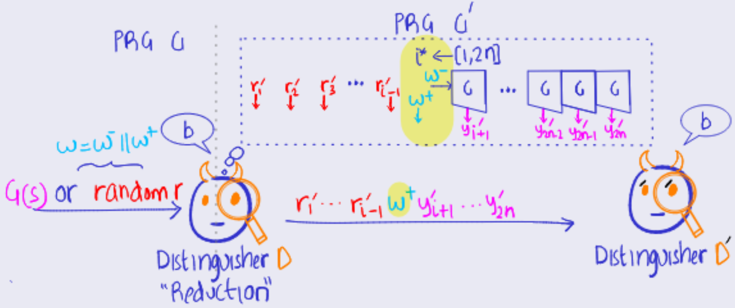
Recall from Previous Lecture...

Theorem 1

If G is a PRG, then so is G′.

Proof. ∃ distinguisher D for G ⇐ ∃ distinguisher D′ for G′.



Analysis: similar to claims 1 and 2.

$\Pr[D'(H_0)=0] - \Pr[D'(H_{2n})=0] \geq \delta \Rightarrow$ $\qquad \Pr[i^*=i] = 1/2n$

$\exists i \in [0, 2n-1]$ such that $\Pr[D'(H_i)=0] - \Pr[D'(H_{i+1})=0] \geq \delta/2n$

Let's Take Stock of Theorem 1

- Construction and Theorem 1 work for any polynomial stretch
  - What happens if we stretch it *exponentially*?

- There is also a "loss in pseudorandomness"
  - D′ distinguishes with some probability $1/p(n) \Rightarrow$
    D distinguishes with probability only $\approx 2n/p(n)$
  - More the stretch, greater the loss

- More generally: "loss in security" of a security reduction
  - One way to measure how "wasteful" the reduction is

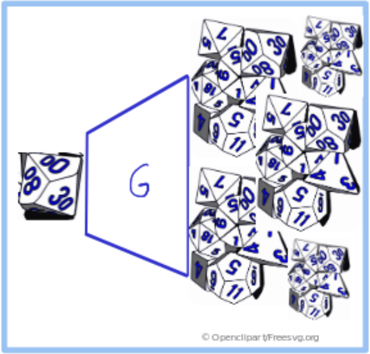
Exercise 1

- Think of a less wasteful reduction strategy for Theorem 1. Do you feel it is possible?
- Maybe need a different construction?

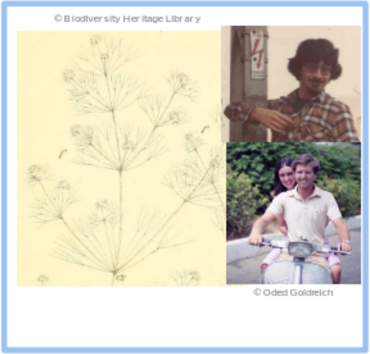
Plan for Today's Lecture

- Task: secure communication of *multiple messages* with shared keys
- Threat model: computational secrecy against eavesdroppers

NEW Pseudo-Random Function (PRF)



NEW GGM PRF

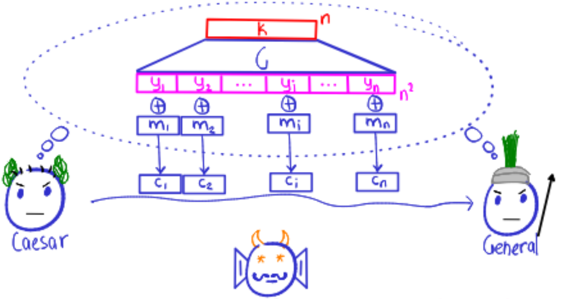


Let's Encrypt Many Messages Using PRG G

$= n$

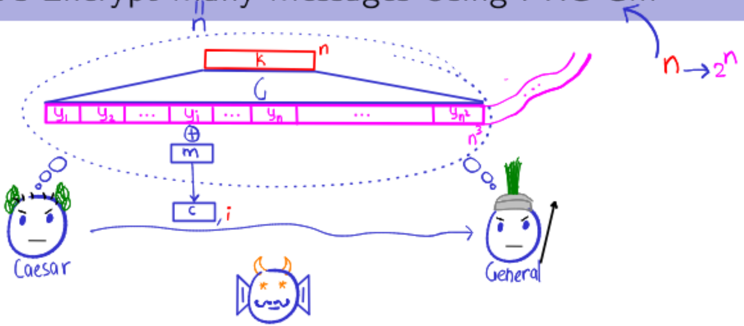
$n \to n^2$

- Setting: Caesar and his general share a key $k \in \{0,1\}^n$ and want to secretly communicate n messages from $\{0,1\}^n$ in presence of eavesdropper Eve*



- SKE construction: use output of G as n pseudorandom OTPs
- Problem: construction *stateful*; synchrony must be maintained
  - We lose correctness if (e.g.) ciphertexts delivered out of order
  - ? Come up with a scenario that leads to loss of secrecy

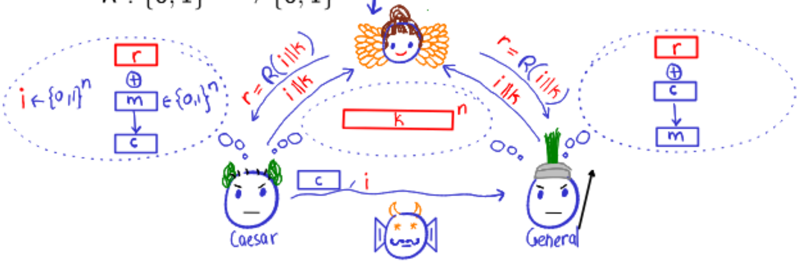
Let's Encrypt Many Messages Using PRG G...



- What if the stretch is n^3? Use OTP at random index $i \in [1, n^2]$
  - Problem? Collision
- Underlying problem: only poly. pseudorandom OTPs available
- What if we stretch the PRG exponentially?
  - Not all pseudorandom OTPs are efficiently "accessible"
- Need "PRG" with
  1. Exponential stretch
  2. Output bits "efficiently" accessible (also called locality)

Let's Encrypt Many Messages Using an Oracle in the Sky

- Setting:
  - Caesar and his general have shared a key $k \in \{0,1\}^n$
  - Everyone (including Eve*) has access to a *random function oracle* $R : \{0,1\}^{2n} \to \{0,1\}^n$



How will you construct a stateless encryption scheme given R?

Hint: R helps generate *exponentially-many* random OTPs

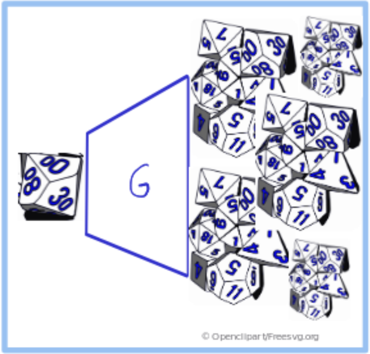
Exercise 2

What if Caesar and his general did not have the shared key k? Can they still do something given the oracle in the sky?

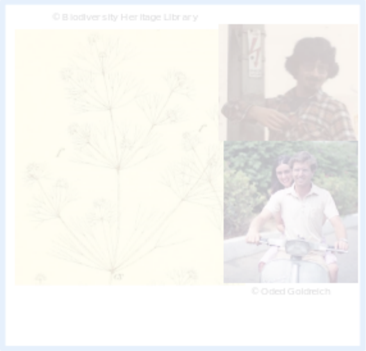
Plan for Today's Lecture

- Task: secure communication of *multiple messages* with shared keys
- Threat model: computational secrecy against eavesdroppers

Pseudo-Random Function (PRF)



GGM PRF



How Exactly to Define Pseudorandomess for Functions? ...

- A function F that "seems like" random function oracle to PPT distinguishers

How Exactly to Define Pseudorandomess for Functions?...

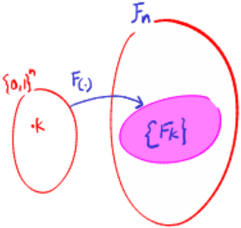
Definition 1 (Two worlds)

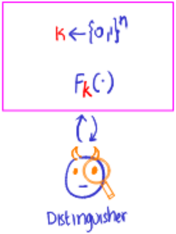
A family of functions $\{F_k : \{0,1\}^n \rightarrow \{0,1\}^n\}_{k \in \{0,1\}^n}$ is a PRF if for every PPT *oracle distinguisher* D

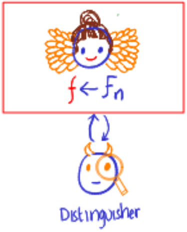
$$\delta(n) := \left| \Pr_{k \leftarrow \{0,1\}^n}[D^{F_k(\cdot)}(1^n) = 0] - \Pr_{f \leftarrow \mathcal{F}_n}[D^{f(\cdot)}(1^n) = 0] \right|$$

pseudorandom world   random world

is negligible.







Let's Check if You Understood Definition 1

PRF or not? Below $F^{(1)}$ and $F^{(2)}$ are PRFs

1. $F_k(x) := k \oplus x$
2. $F_{k_1 \| k_2}(x) := F^{(1)}_{k_1}(x) \| F^{(2)}_{k_2}(x)$
3. $F_k(x_1 \| x_2) := F^{(1)}_k(x_1) \| F^{(2)}_k(x_2)$

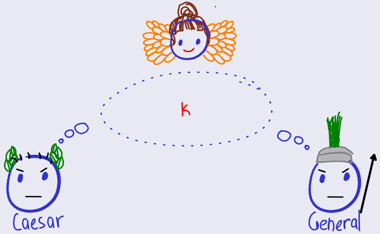
PRG or not? Below, F is a PRF

1. $G(s) := F_s(1) \| F_s(2) \| \cdots \| F_s(n-1) \| F_s(n)$
2. $G(s) := F_s(2^0) \| F_s(2^1) \cdots \| F_s(2^{n-1}) \| F_s(2^n)$
3. $G(s) := F_1(s) \| F_2(s) \| \cdots \| F_{n-1}(s) \| F_n(s)$

Exercise 3

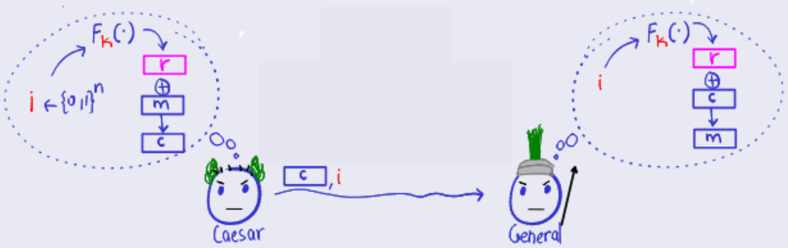
In all the "yes" cases above, formally prove; in all the "no" cases, describe a counter-example.

Construction 1 (Replace random oracle with PRF)



(Stateless) Symmetric-Key Encryption from PRF

Construction 1 (Replace random oracle with PRF)



Note: encryption is randomised and thus length of ciphertext is longer than plaintext (first such scheme in this course)

Exercise 4 (Hint: reduction similar to computational OTP)

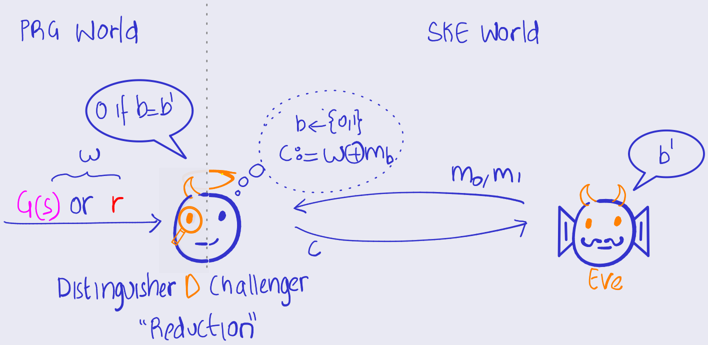
1. Formulate the eavesdropper threat model for multiple encryptions
2. Prove that Construction 1 is secure against eavesdroppers

Hint: Reduction Similar to Computational OTP...

Theorem 2 (Recall, Lecture 05-06)

If G is a PRG, then Comp. OTP is comp. secret against eavesdroppers

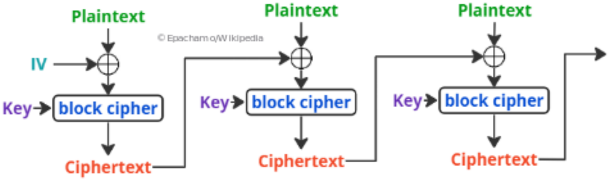
Proof by reduction. $\exists$D for G $\Leftarrow$ $\exists$Eve breaking Computational OTP.



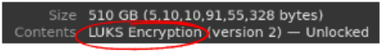
□

PRFs IRL

- Coming up: theoretical construction, but inefficient for practice
- Practical PRFs: block ciphers like AES
  - Usually only support certain key-sizes (128, 192, 256)
  - Supported by most libraries (e.g., OpenSSL, NaCl) and even implemented on modern processors (AES-NI)
- For encrypting larger messages (e.g., for disk encryption) "modes of operation" used (Coming up in Lecture 08!)
  - E.g: Cipher block-chaining (CBC) mode



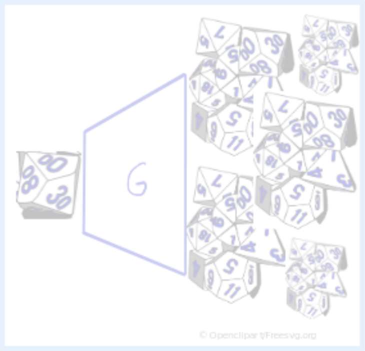
  - My laptop uses LUKS for disk encryption, which uses AES-XTS



Plan for Today's Lecture

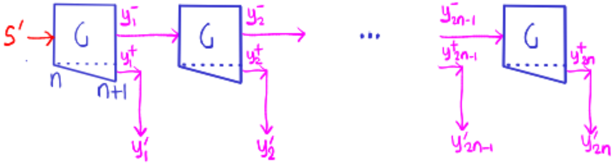
- Task: secure communication of *multiple messages* with shared keys
- Threat model: computational secrecy against eavesdroppers

NEW Pseudo-Random Function (PRF)



NEW GGM PRF

Let's Try to Construct a PRF

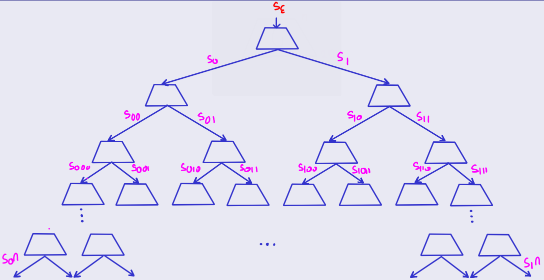


- Recall construction of length-extending PRG from last lecture
- Recall the problem with expanding exponentially:
  - Takes exponential time to access most pseudorandom OTPs
- Need "PRG" with
  1. Exponential stretch
  2. Output bits "efficiently" accessible (also called locality)
- How to reconcile the two requirements?
  - Hint: Use length-doubling PRG
  - Use binary tree instead of chain!

Tree-Based Construction from Length-Doubling PRG G

$n \to n^2$

Construction 2 (GGM PRF $\{F_k : \{0,1\}^n \to \{0,1\}^n\}_{k\in\{0,1\}^n}$)



- Define $F_k(x) = s_x$ with $s_\varepsilon := k$

Exercise 5

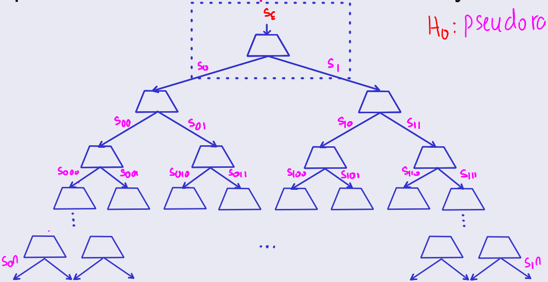
1. Write down the construction formally.
2. What if we use d-ary tree instead of binary tree?

Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random



H_0: pseudorandom world

Theorem 3

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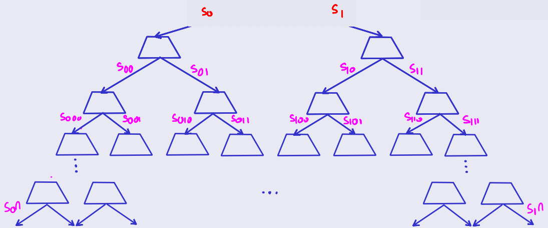
Strategy: replace, breadth-first, pseudorandom by random

H_1: hybrid world



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How do We Prove that Construction 2 is a PRF?...

Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

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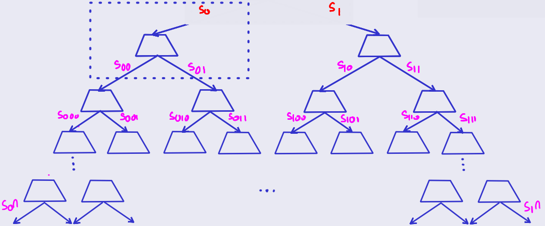
H_1: hybrid world



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How do We Prove that Construction 2 is a PRF?...

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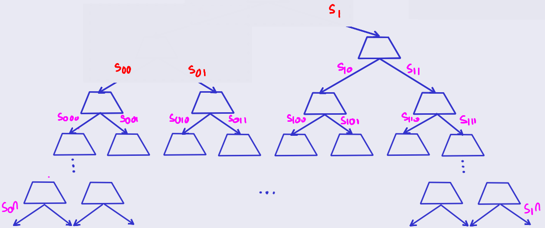
Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random

H_2: hybrid world



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How do We Prove that Construction 2 is a PRF?

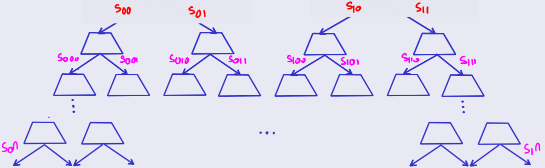
Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random

H_3: hybrid world



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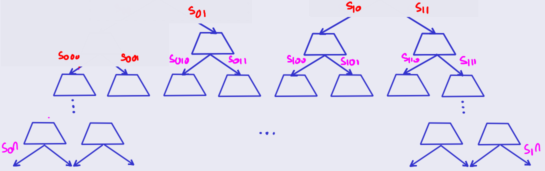
Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random

H_3: hybrid world



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How do We Prove that Construction 2 is a PRF?...

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How do We Prove that Construction 2 is a PRF?...

Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random

$H_{2^{n+1}}$: random world

s_{0^n} ... s_{1^n}

How do We Prove that Construction 2 is a PRF?

Theorem 3

If G is a length-doubling PRG, then Construction 2 is a PRF.

Proof. First attempt: off-the-shelf hybrid argument.

Strategy: replace, breadth-first, pseudorandom by random

Hybrid argument: if D can distinguish H_0 from $H_{2^{n+1}}$ w/ pr. δ

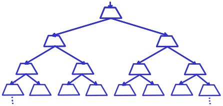
$\Downarrow$

$\exists i \in [0, 2^{n+1}-1]$ such that D distinguishes H_i from H_{i+1} w/ pr. $\delta/2^{n+1}$

Problem: exponential number of hybrids

Solution: hybrid argument with on-the-fly/lazy sampling!

Recap/Next Lecture



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- Defined and constructed PRFs
  - GGM tree-based construction from length-doubling PRGs
  - Another application of hybrid argument

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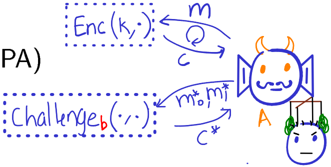

- Constructed a *stateless* SKE from PRF
  - Next lecture: chosen-plaintext attack (CPA)
- Other applications of PRFs
  - Authentication (coming up: Lecture 09)
  - Natural proofs: barrier to resolving the P $\stackrel{?}{=}$ NP question

Further Reading

1. PRFs were introduced in [GGM84], where the namesake construction from PRGs was also presented.
2. [Gol01, §3.6] for a formal proof of Theorem 3
3. [KL14, §3.5] for a formal description of Construction 1.
4. To read more about natural proofs, and the role of PRFs there [Aar03, §4] or [Cho11] are good sources.

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