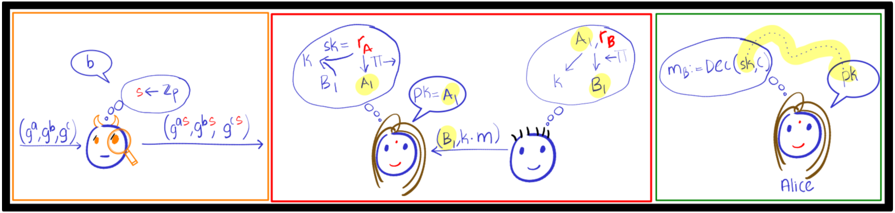
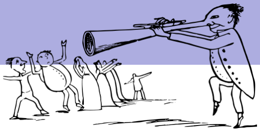


CS409m: Introduction to Cryptography

Lecture 13 (26/Sep/25)

Instructor: Chethan Kamath

Announcements



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Changes to mid-sem crib session

- View your answer sheet 12:30-14:30 on Monday (29/Sep) in CC305
- Submit cribs online by Wednesday (01/Oct, 23:59)

Bounty on Problem 7.3:

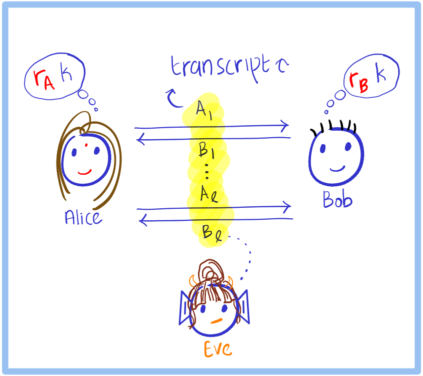
- Come up with a *simple* construction of MAC from weak PRF
- Construction provided in solution set is too complex!

- Quiz 2 on 08/Oct, 08:25-09:25

Recall from Last Lecture...

- Task: key exchange
- Threat model: computational secrecy against eavesdroppers

Key Exchange



Basic Group Theory

Definition 3 (Lecture 11)

An Abelian group $\mathbb{G}$ is a set $\mathcal{G}$ with a binary op. $\cdot$ satisfying:

1. Closure
2. Associativity
3. Existence of identity
4. Existence of inverse
5. Commutativity

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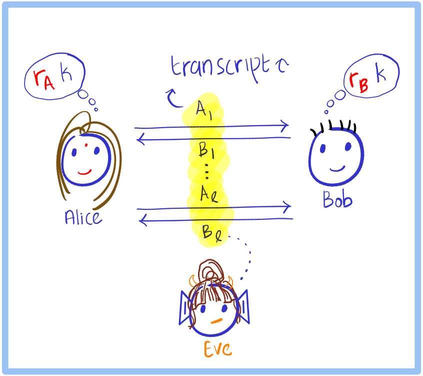

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1. Closure
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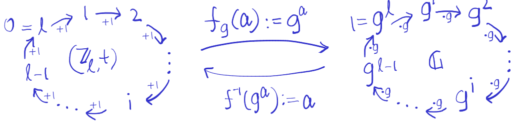
Motivation: need richer algebraic *structure* to construct key exchange

Recall from Last Lecture...

- Cyclic group: there exists a "generator" $g \in \mathcal{G}$ with order $\ell = |\mathcal{G}|$
  - That is $\{g^1 = g, g^2, \ldots, g^{\ell-1}, g^\ell = 1\} = \mathcal{G}$



  - "Isomorphism" between $(\mathbb{Z}_\ell, +)$ and $\mathbb{G}$

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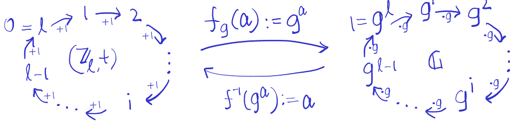
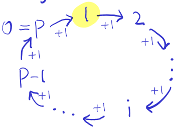


  - "Isomorphism" between $(\mathbb{Z}_\ell, +)$ and $\mathbb{G}$
- Examples:

Addition modulo prime p

◆ order p ◆ cyclic

$(\mathbb{Z}_p, +)$

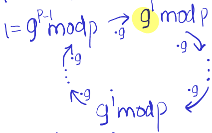


$\{0, \ldots p-1\}$ $\quad g_1 + g_2 := g_1 + g_2 \pmod p$

Multiplication modulo prime p

◆ order $p-1$ ◆ cyclic

$(\mathbb{Z}_p^\times, \cdot)$



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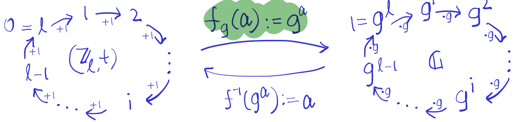

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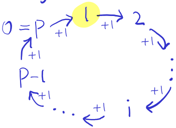	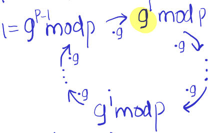
$(\mathbb{Z}_p,+)$	$(\mathbb{Z}_p^\times,\cdot)$
$\{0,\ldots p-1\}$ $g_1+g_2 := g_1+g_2 (\text{mod } p)$	$\{1,\ldots p-1\}$ $g_1\cdot g_2 := g_1\cdot g_2 (\text{mod } p)$

- Easy to compute: Group operation, exponentiation, inverse etc.

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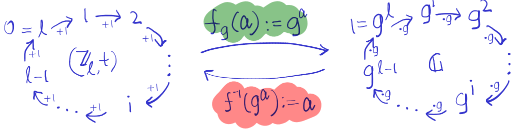

  - "Isomorphism" between $(\mathbb{Z}_\ell, +)$ and $\mathbb{G}$
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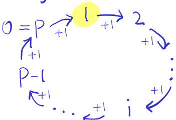
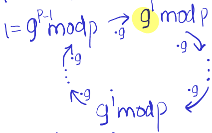
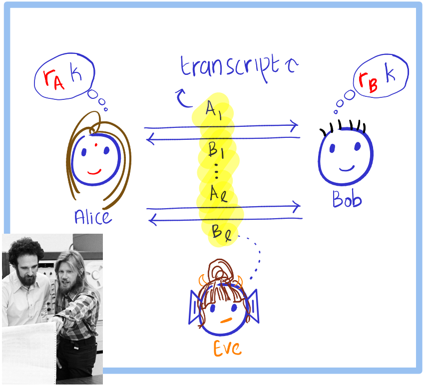
	
$(\mathbb{Z}_p, +)$: $\{0, \ldots, p-1\}$, $g_1 + g_2 := g_1 + g_2 \pmod p$	$(\mathbb{Z}_p^\times, \cdot)$: $\{1, \ldots, p-1\}$, $g_1 \cdot g_2 := g_1 \cdot g_2 \pmod p$

- Easy to compute: Group operation, exponentiation, inverse etc.
- What is possibly hard to compute? Discrete logarithm (DLP)

Plan for Today's Lecture

NEW Diffie-Hellman Key Exchange



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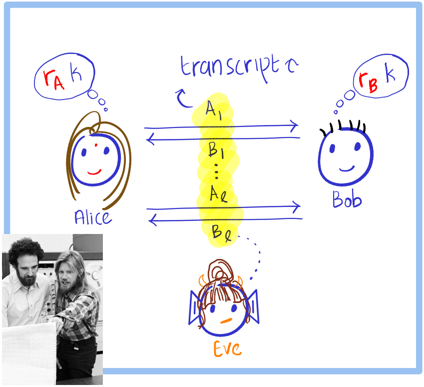
Plan for Today's Lecture

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange



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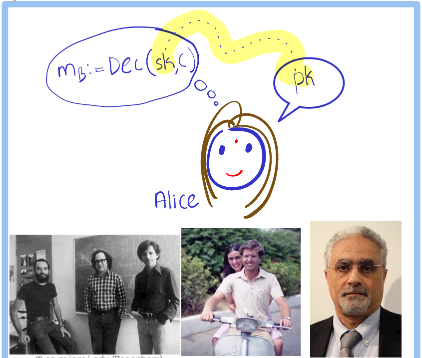
NEW Public-Key Encryption



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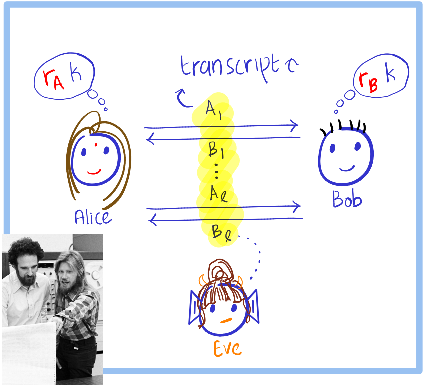

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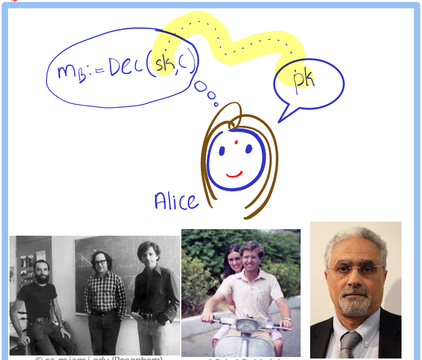
Public-Key Encryption



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Underlying hard problem: Decisional Diffie-Hellman (DDH)

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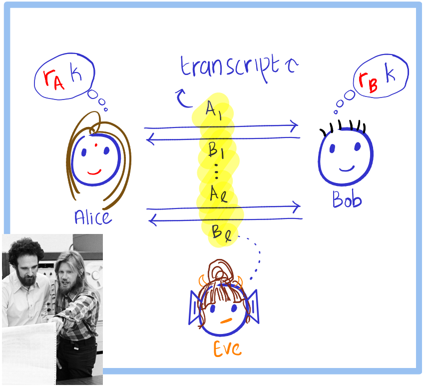

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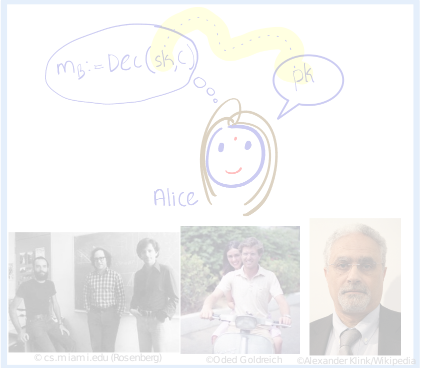
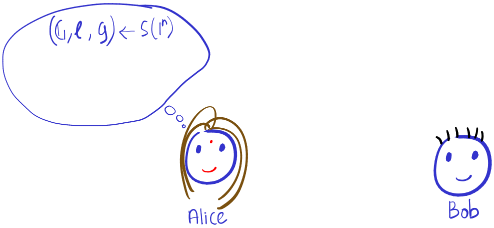


Underlying hard problem: Decisional Diffie-Hellman (DDH)

Diffie-Hellman Key-Exchange Protocol

Protocol 1

Diffie-Hellman Key-Exchange Protocol



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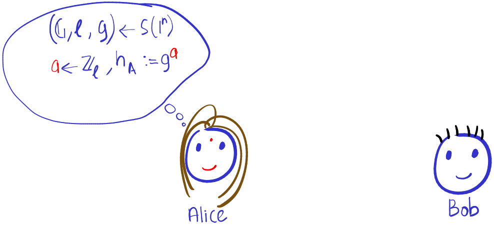

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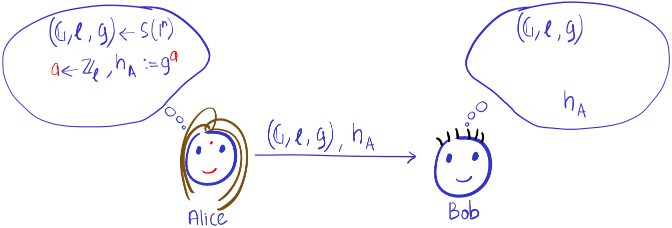

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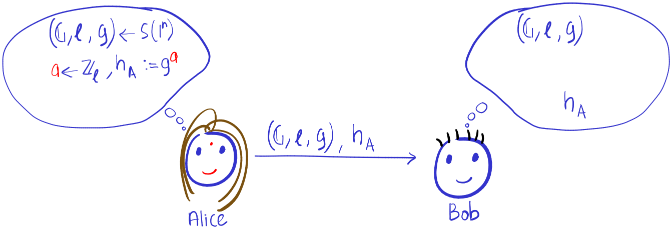

Protocol 1

1. Alice→Bob: Send $((\mathbb{G},\ell,g), h_A := g^a)$, where $(\mathbb{G},\ell,g) \leftarrow S(1^n)$ and $a \leftarrow \mathbb{Z}_\ell$

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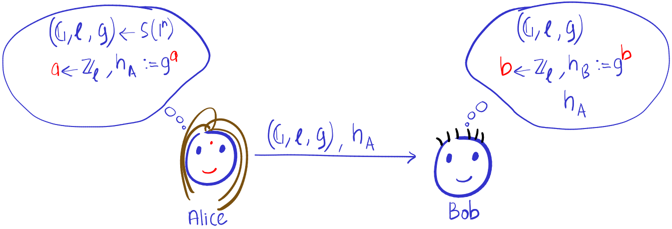

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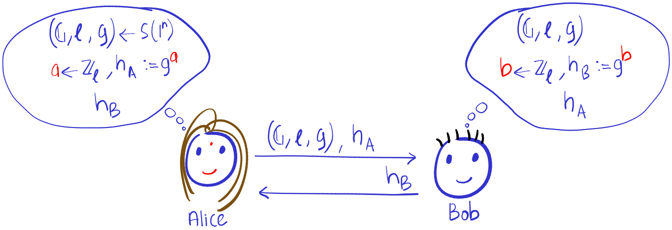

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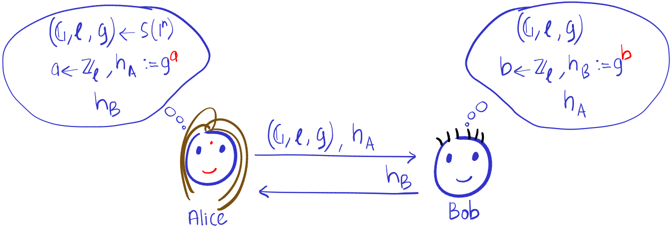

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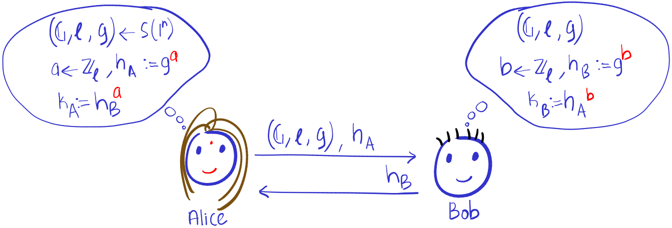

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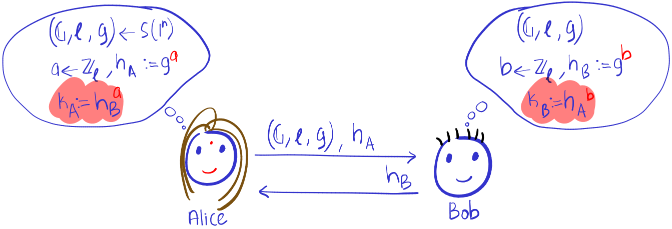

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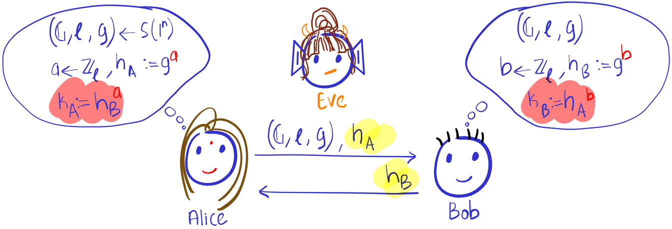
3. Alice outputs $k_A := (h_B)^a$; Bob outputs $k_B := (h_A)^b$

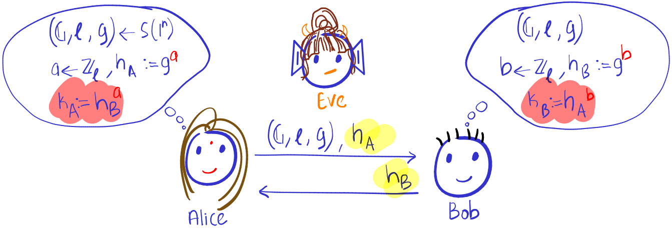
- Correctness of key generation (by Exercise 4, Lecture 12):

$$k_A = h_B^a = (g^b)^a = g^{ab} = (g^a)^b = h_A^b = k_B$$

When is it Secret Against Eavesdroppers?



- What does Eve see? The transcript is $(h_A := g^a, h_B := g^b)$



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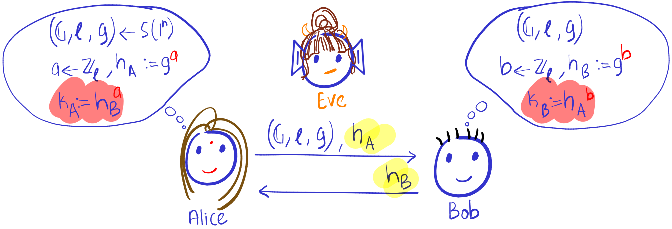

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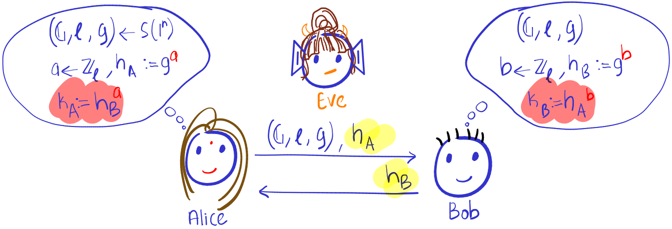

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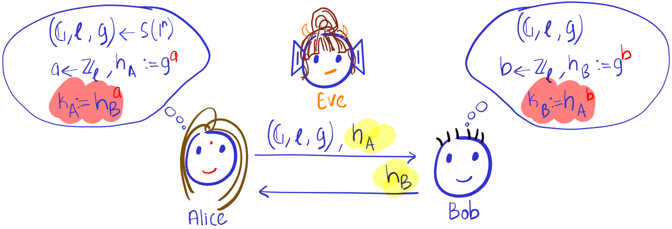

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  - No, what if Eve can compute g^{ab} given g^a and g^b?
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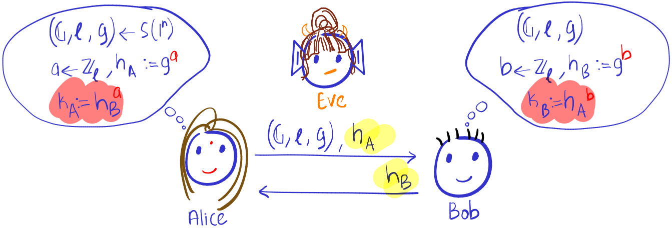

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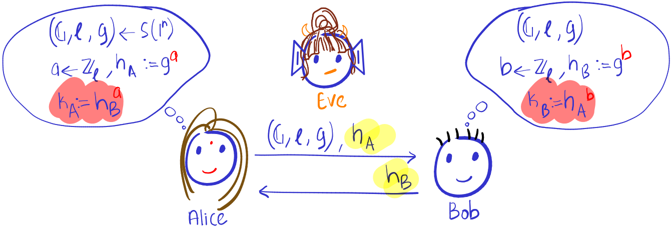

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- Is CDH problem being hard sufficient?
  - What if Eve can distinguish g^{ab} from random group elements?
    - There exist such groups!

When is it Secret Against Eavesdroppers?...

Assumption 1 (Decisional DH (DDH) assumption in $\mathbb{G}$ w.r.to $\mathsf{S}\cdots$)

$\cdots$ holds if for all PPT distinguishers D, the following is negligible:

$$\Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^{ab}) = 0] - \Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b,r\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^{r}) = 0]$$

"Real world" "Random world"

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Diffie-Hellman key-exchange is computationally secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof.

Secrecy requirement is same as the assumption! □

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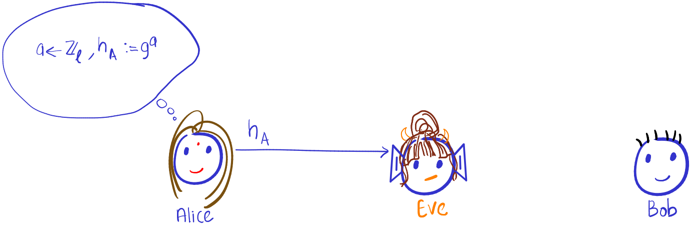
Exercise 1

But I did slightly cheat! Figure out where.

What About Secrecy Against Active Eve?



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    - Recall that active Eve can intercept/tamper messages

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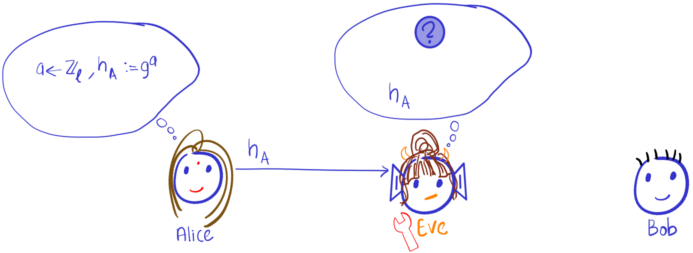

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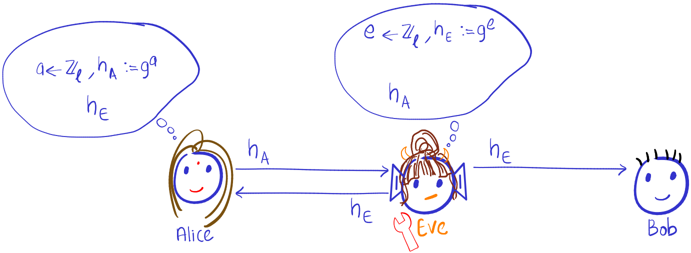

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⚠ There is a person-in-the-middle attack!

  - Pretends to be Alice to Bob and Bob to Alice
  - Eve sets up two separate key exchanges with Alice and Bob

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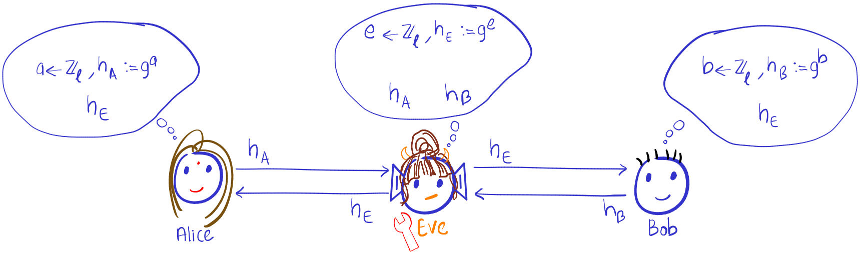

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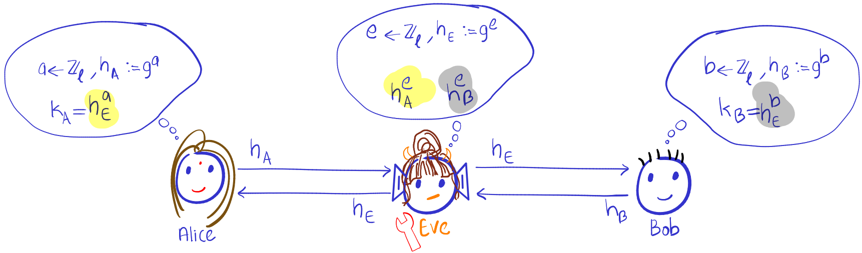

- What if Eve is an *active* adversary?
  - Recall that active Eve can intercept/tamper messages

⚠ There is a person-in-the-middle attack!

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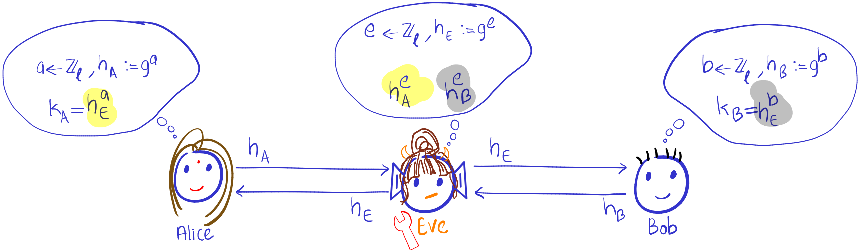

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⚠ Insecure against active adversary

Where is DDH Assumption Known to Hold?

Addition modulo prime p

$0 = p \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} \cdots \xrightarrow{+1} i \xrightarrow{+1} \cdots \xrightarrow{+1} p-1 \xrightarrow{+1} 0$

$(\mathbb{Z}_p, +)$

$\{0, \dots, p-1\}$ $\qquad g_1 + g_2 := g_1 + g_2 \pmod p$

- order p
- cyclic

Multiplication modulo prime p

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- order $p-1$
- cyclic

Multiplication modulo $N = pq$

primes

$(\mathbb{Z}_N^\times, \cdot)$

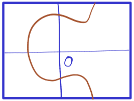
$\{1, \dots, p-1, p+1, \dots, 2p-1, \dots, pq-1\}$ $\qquad g_1 \cdot g_2 := g_1 \cdot g_2 \pmod N$

- order $(p-1)(q-1)$
- not cyclic

Elliptic curves modulo prime p

$(E, +)$ "curve points addition"

solutions to $y^2 = x^3 + Ax + B \pmod p$



- $|p+1 - \text{order}| \leq 2\sqrt{p}$
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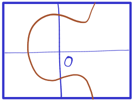

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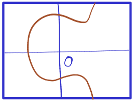

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◆ order $p-1$ ◆ cyclic ✓ Hard in subgroup

Multiplication modulo $N = pq$

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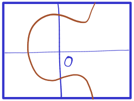

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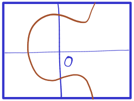

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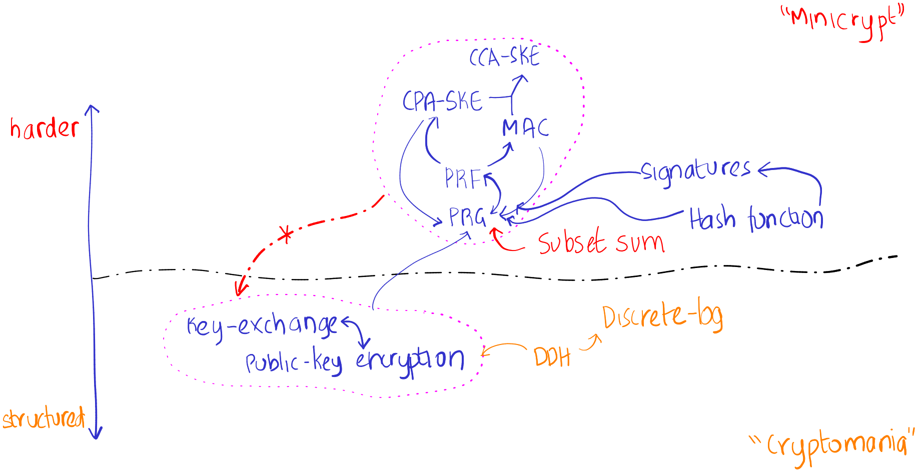
Believed hard

What Else Can be Built from DDH?



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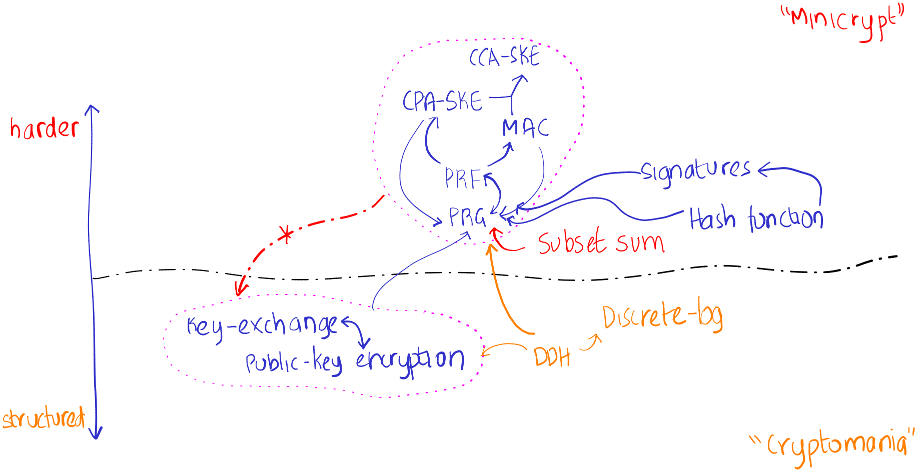


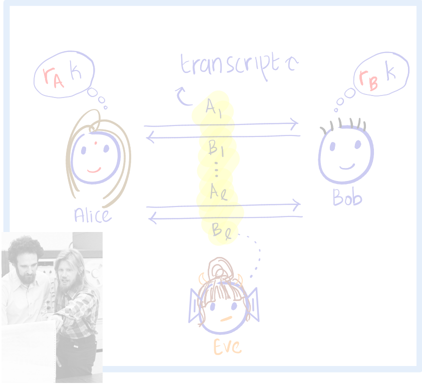
Exercise 2

Construct a PRG from DDH

Plan for Today's Lecture

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange



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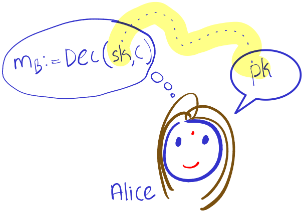
NEW Public-Key Encryption



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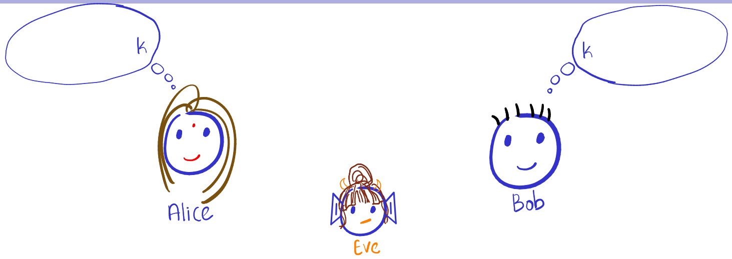
Underlying hard problem: Decisional Diffie-Hellman (DDH)



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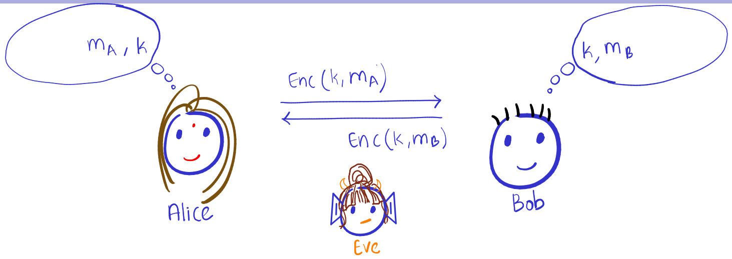

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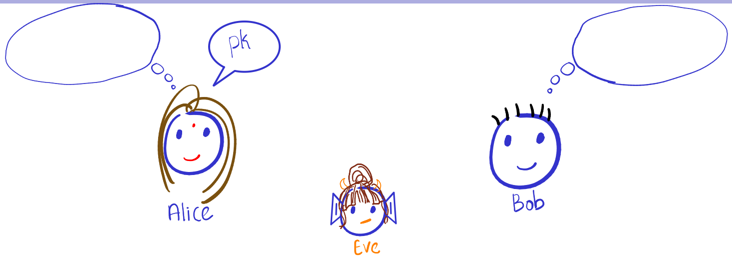
The Setting: Shared (Private) Keys vs Public Keys...



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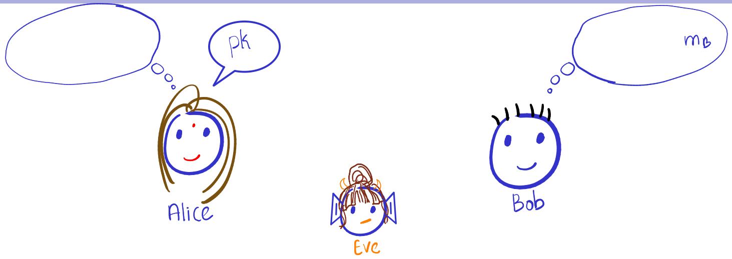

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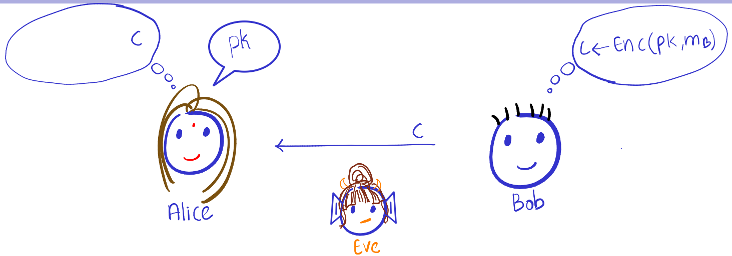

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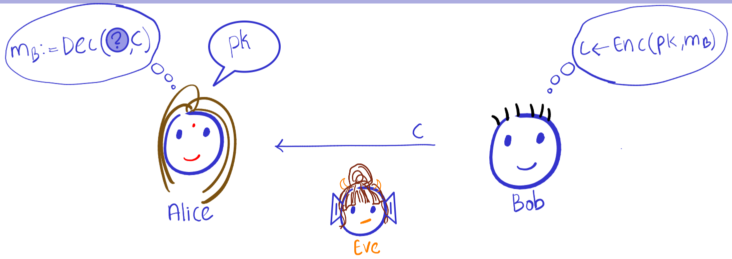

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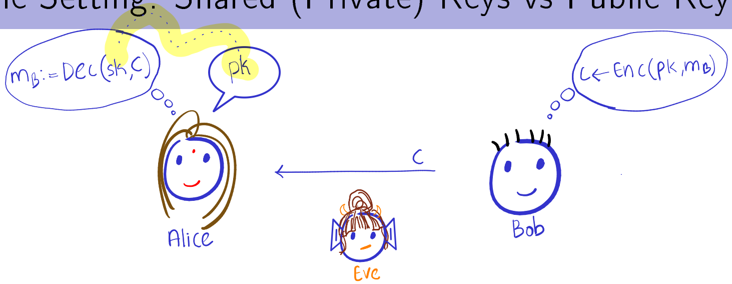

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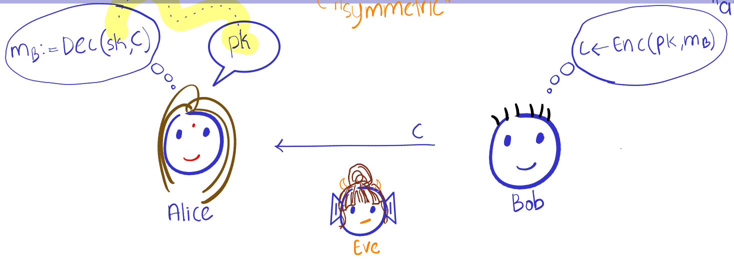
"symmetric" "asymmetric"



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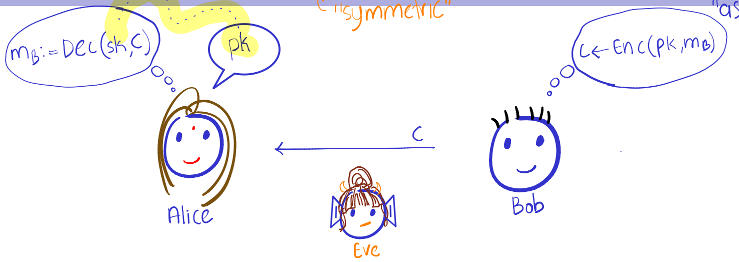

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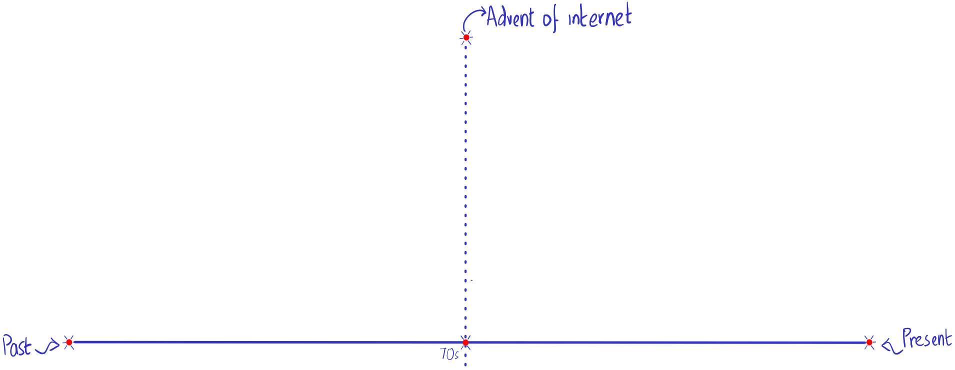
+ Advantage: scalability! It suffices to have one "key" per user

The Setting: Shared (Private) Keys vs Public Keys...



The Setting: Shared (Private) Keys vs Public Keys...

Advent of internet

Programming Techniques — S. L. Graham, R. L. Rivest, Editors

Secure Communications Over Insecure Channels

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DHKE

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RSA cryptosystem

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Past 70s 74 76 78 Present

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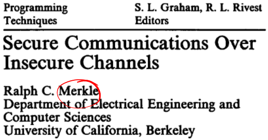
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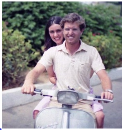
Goldwasser–Micali PKE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Past — 70s — 74 — 76 — 78 — 82 — Present

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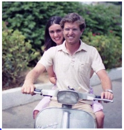

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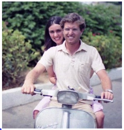

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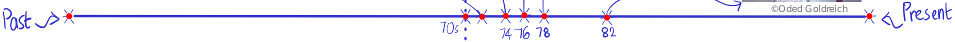
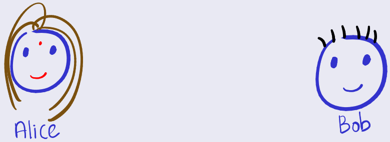


- PKE IRL: PGP, hybrid encryption

Syntax of Public-Key Encryption

Definition 4 (Public-Key Encryption (PKE))

A PKE Π is a triple of efficient algorithms $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ with the following syntax:



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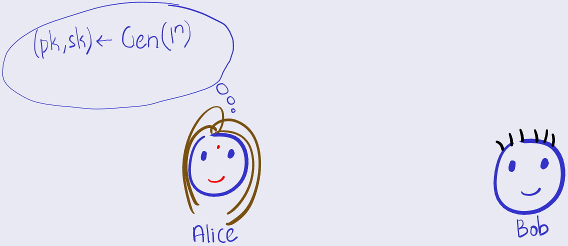

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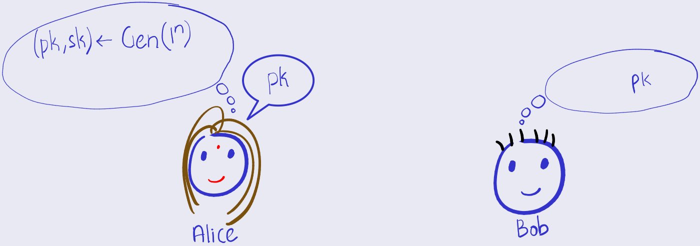

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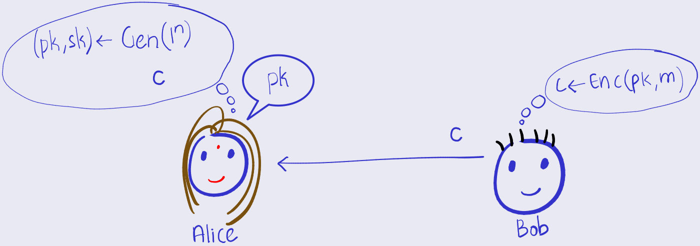

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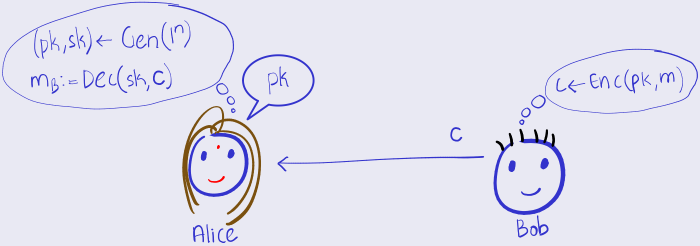

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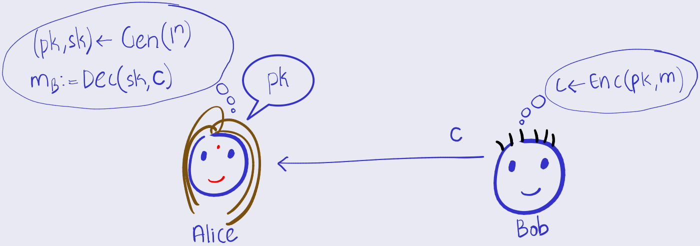

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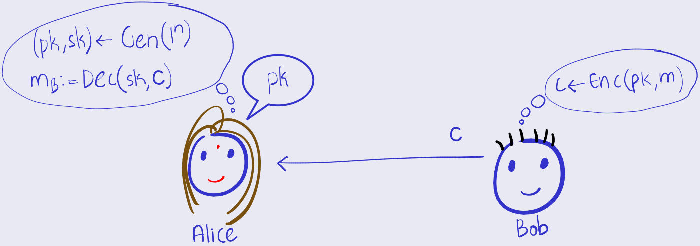

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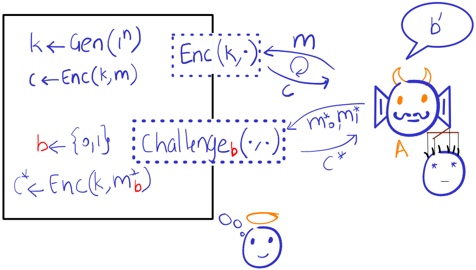
? How can an unbounded eavesdropper Eve break PKE?

How to Define Security?

- Recall CPA-secrecy requirement in the SKE setting



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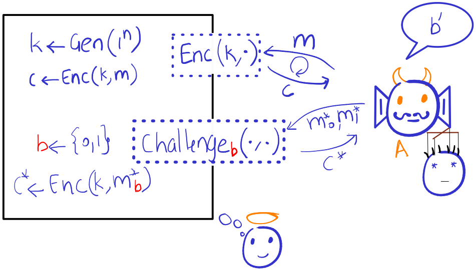

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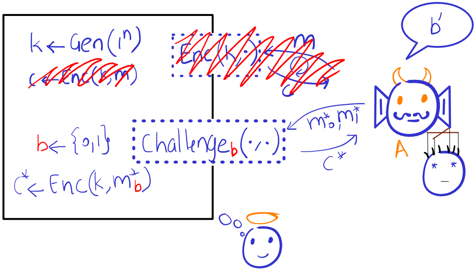

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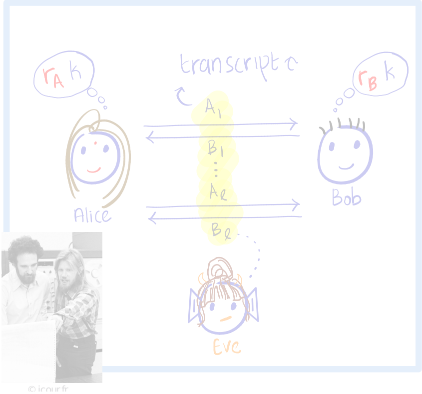
- Stronger notion: ind. against chosen-ciphertext attack (CCA)

Plan for Today's Lecture

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange



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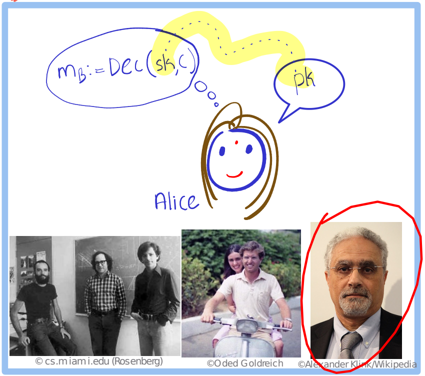
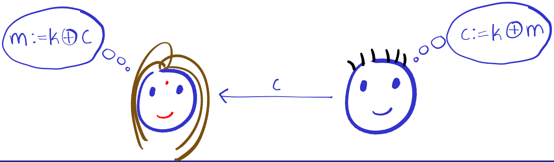


Underlying hard problem: Decisional Diffie-Hellman (DDH)

One-Time Pad Can be Generalised Over Groups



Pseudocode 1 (OTP over $(\{0,1\}^n, \oplus)$ with message space $\{0,1\}^n$)

- Key generation Gen: output $k \leftarrow \{0,1\}^n$
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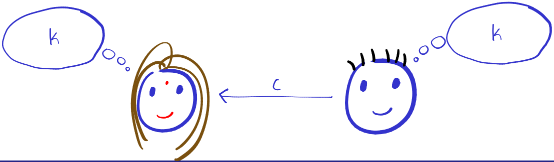

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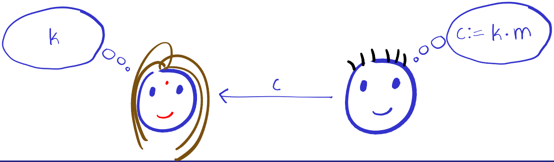

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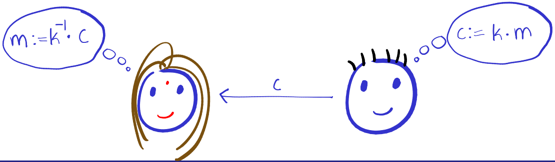

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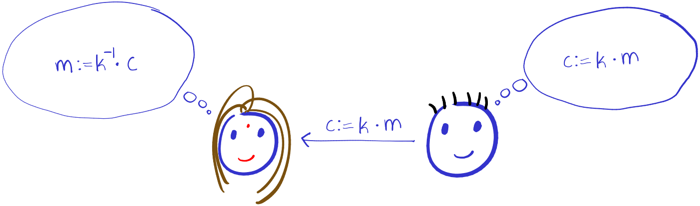

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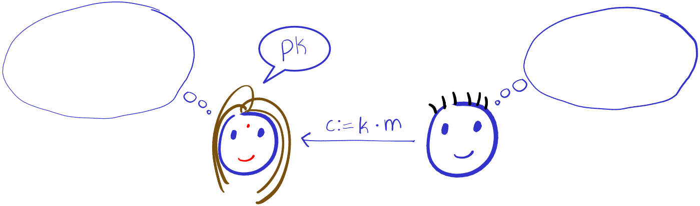

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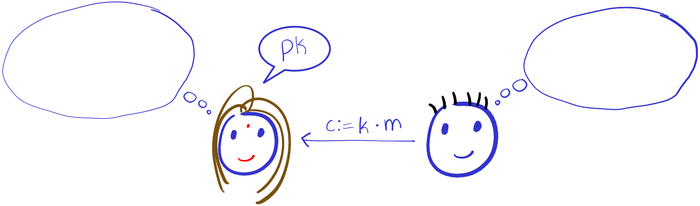

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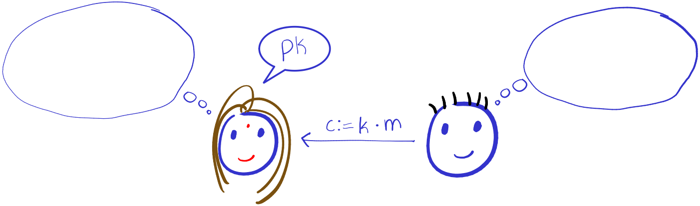

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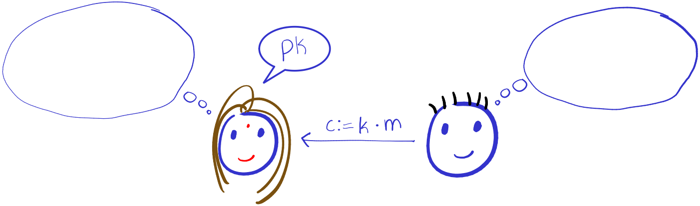

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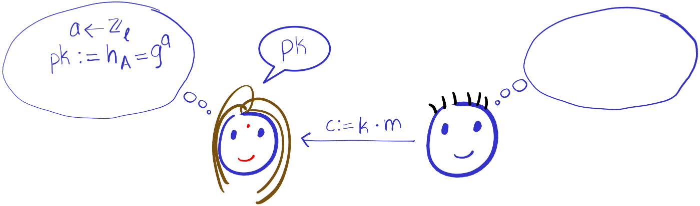

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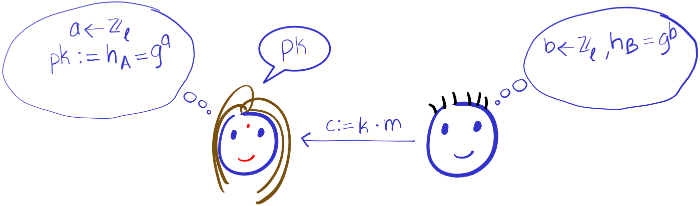

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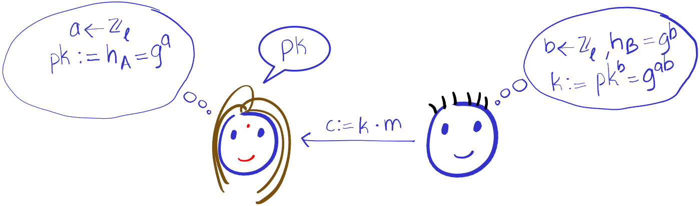

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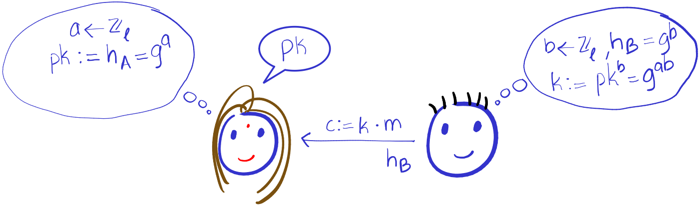

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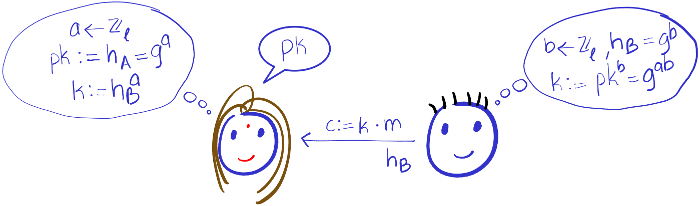

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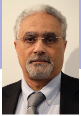

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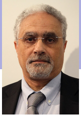

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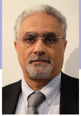

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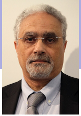

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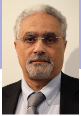

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? Why is H_0' indistinguishable from H_1'? OTP over group

Is it a Coincidence that ElGamal is Similar to DHKE? No!

Claim 1 (Two-message KE $\rightarrow$ CPA-PKE)

If two-message key exchange protocol Π exists then so does PKE.

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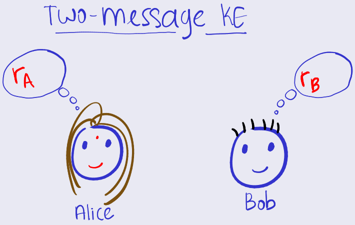

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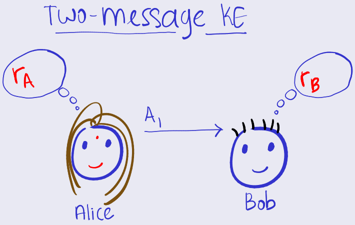

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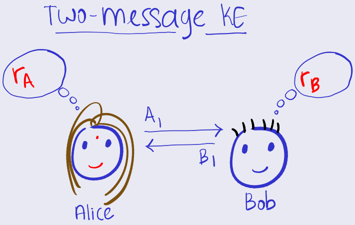

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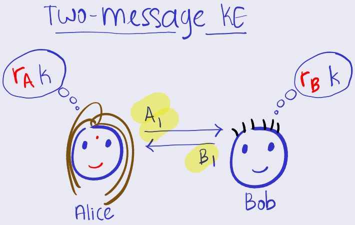

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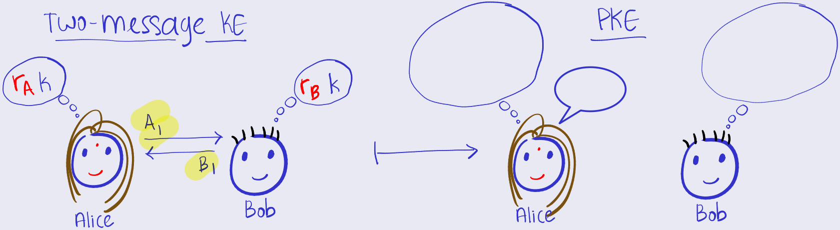

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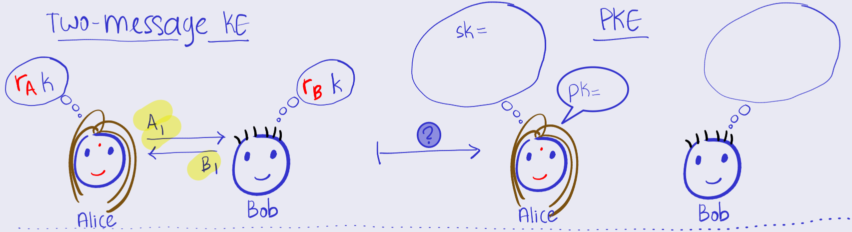

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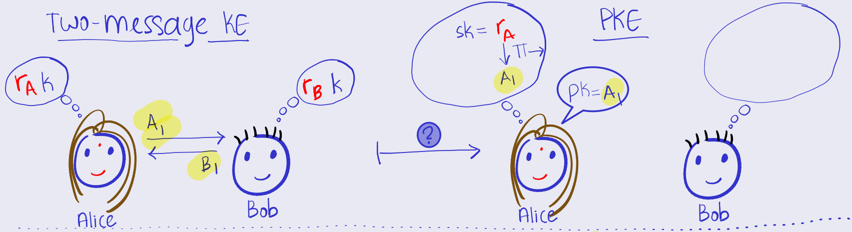

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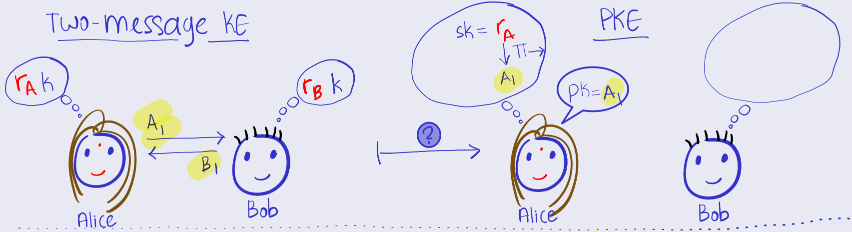

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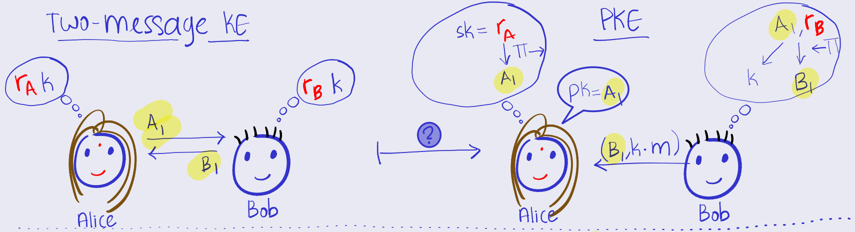

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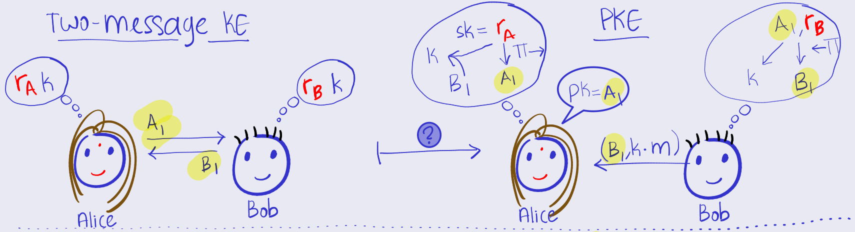

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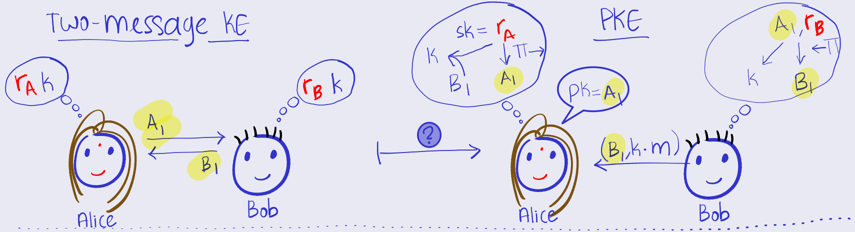

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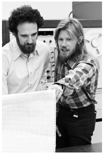
Exercise 3 (Converse to Claim 1: two-message KE $\leftarrow$ CPA-PKE)

If PKE exists then so does two-message key exchange.

Recap/Next Lecture



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- Diffie-Hellman key exchange (DHKE)
  - Based on DDH assumption in cyclic groups
  - Algebraic *structure* exploited: $(g^a)^b = g^{ab} = (g^b)^a$

Recap/Next Lecture

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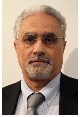

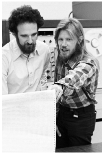

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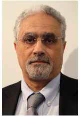

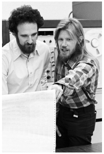

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- Next lecture:
  - Factoring and related hardness assumptions
  - RSA group: multiplicative group modulo $N := pq$
  - Goldwasser-Micali encryption
  - RSA encryption

$a \leftarrow \mathbb{Z}_\ell$
$pk := h_A = g^a$
$k := h_B^a$

pk

$c := k \cdot m$

h_B

Key-exchange

Public-key encryption

DDH

Factoring

QR

References

1. [KL14, Chapter 11] for more details on key exchange
2. Read the seminal paper by Diffie and Hellman [DH76] for a description of the namesake key-exchange. In general this paper is a very insightful read.
3. Boneh's survey [Bon98] is an excellent source on the DDH problem.

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Whitfield Diffie and Martin E. Hellman.
New directions in cryptography.
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Introduction to Modern Cryptography (3rd ed.).
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