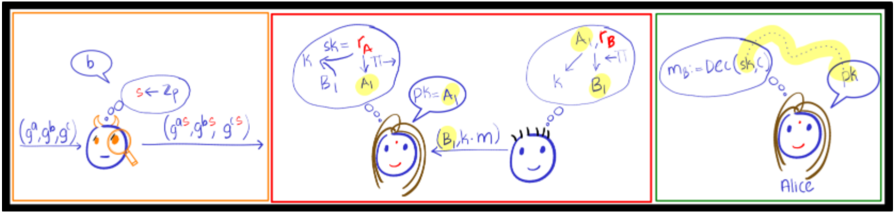

CS409m: Introduction to Cryptography

Lecture 13 (26/Sep/25)

Instructor: Chethan Kamath

Announcements

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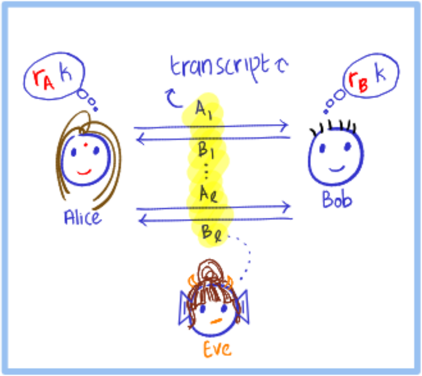
- △ Changes to mid-sem crib session
  - View your answer sheet 12:30-14:30 on Monday (29/Sep) in CC305
  - Submit cribs online by Wednesday (01/Oct, 23:59)
- ⚠ Bounty on Problem 7.3:
  - Come up with a *simple* construction of MAC from weak PRF
  - Construction provided in solution set is too complex!

- Quiz 2 on 08/Oct, 08:25-09:25

Recall from Last Lecture

- Task: key exchange
- Threat model: computational secrecy against eavesdroppers

Key Exchange



Basic Group Theory

Definition 3 (Lecture 11)

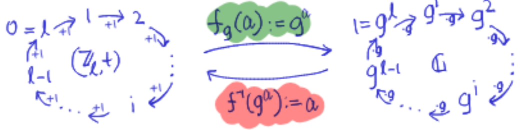
An Abelian group $\mathbb{G}$ is a set $\mathcal{G}$ with a binary op. $\cdot$ satisfying:

1. Closure
2. Associativity
3. Existence of identity
4. Existence of inverse
5. Commutativity

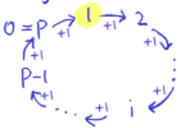
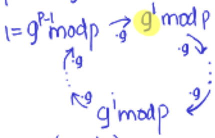
Motivation: need richer algebraic *structure* to construct key exchange

Recall from Last Lecture...

- **Cyclic group**: there exists a "generator" $g \in \mathcal{G}$ with order $\ell = |\mathcal{G}|$
  - That is $\{g^1 = g, g^2, \ldots, g^{\ell-1}, g^\ell = 1\} = \mathcal{G}$



  - "Isomorphism" between $(\mathbb{Z}_\ell, +)$ and $\mathbb{G}$
- Examples:

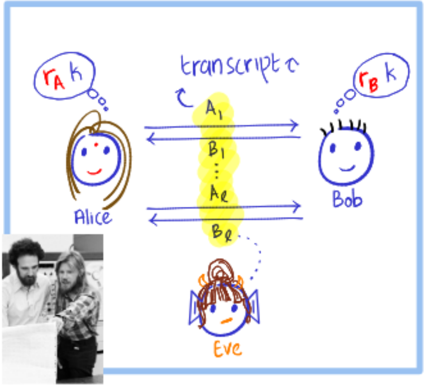
Addition modulo prime p	Multiplication modulo prime p
◆ order p ◆ cyclic	◆ order $p-1$ ◆ cyclic
$(\mathbb{Z}_p,+)$: $\{0,\cdots p-1\}$, $g_1+g_2 := g_1+g_2 (\mathrm{mod}\, p)$	$(\mathbb{Z}_p^\times,\cdot)$: $\{1,\cdots p-1\}$, $g_1 \cdot g_2 := g_1 \cdot g_2 (\mathrm{mod}\, p)$
	

- **Easy** to compute: Group operation, exponentiation, inverse etc.
- What is possibly **hard** to compute? Discrete logarithm (DLP)

Plan for Today's Lecture

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange



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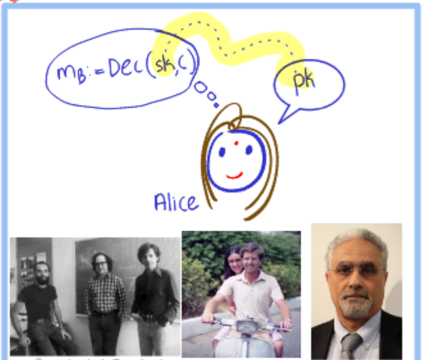
NEW Public-Key Encryption



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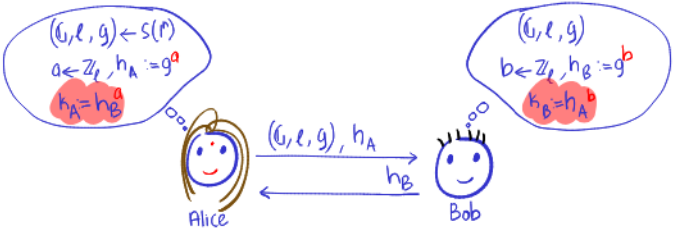
Underlying hard problem: Decisional Diffie-Hellman (DDH)

Diffie-Hellman Key-Exchange Protocol



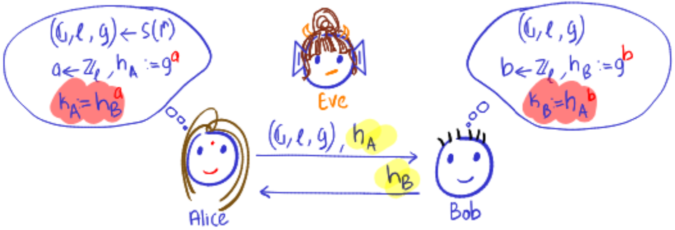
Protocol 1

1. Alice→Bob: Send $((\mathbb{G}, \ell, g), h_A := g^a)$, where $(\mathbb{G}, \ell, g) \leftarrow \mathsf{S}(1^n)$ and $a \leftarrow \mathbb{Z}_\ell$
2. Alice←Bob: Send $h_B := g^b$ for $b \leftarrow \mathbb{Z}_\ell$
3. Alice outputs $k_A := (h_B)^a$; Bob outputs $k_B := (h_A)^b$

- Correctness of key generation (by Exercise 4, Lecture 12):

$$k_A = h_B^a = (g^b)^a = g^{ab} = (g^a)^b = h_A^b = k_B$$

When is it Secret Against Eavesdroppers?



- What does Eve see? The transcript is $(h_A := g^a, h_B := g^b)$
- What if DLog problem is easy over $\mathbb{G}$?
  - Then Eve can invert h_A to get a and compute $k = h_B^a$
- Is DLog problem being hard sufficient?
  - No, what if Eve can compute g^{ab} given g^a and g^b?
    - This is the "computational Diffie-Hellman" (CDH) problem
- Is CDH problem being hard sufficient?
  - What if Eve can distinguish g^{ab} from random group elements?
    - There exist such groups!

When is it Secret Against Eavesdroppers?...

Assumption 1 (Decisional DH (DDH) assumption in $\mathbb{G}$ w.r.to $\mathsf{S}\cdots$)

$\cdots$ holds if for all PPT distinguishers D, the following is negligible:

$$\Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^{ab}) = 0] - \Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b,r\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^r) = 0]$$

"Real world" "Random world"

Theorem 1

Diffie-Hellman key-exchange is computationally secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof.

Secrecy requirement is same as the assumption! $\square$

Exercise 1

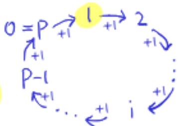
But I did slightly cheat! Figure out where.

- What if Eve is an *active* adversary?
  - Recall that active Eve can intercept/tamper messages
- ⚠ There is a person-in-the-middle attack!
  - Pretends to be Alice to Bob and Bob to Alice
  - Eve sets up two separate key exchanges with Alice and Bob
- ⚠ Insecure against active adversary

Where is DDH Assumption Known to Hold?

⚠ Easy! Since DLog is easy

Addition modulo prime p



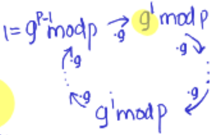
$(\mathbb{Z}_p, +)$

$\{0, \dots p-1\}$ $\quad g_1 + g_2 := g_1 + g_2 \pmod p$

◆ order p ◆ cyclic

⚠ Easy! See Assign. 4.

Multiplication modulo prime p



$(\mathbb{Z}_p^\times, \cdot)$

$\{1, \dots p-1\}$ $\quad g_1 \cdot g_2 := g_1 \cdot g_2 \pmod p$

◆ order $p-1$ ◆ cyclic ✓

Hard in subgroup

Multiplication modulo $N = pq$

primes

$(\mathbb{Z}_N^\times, \cdot)$

$\{1, \dots p-1, p+1, \dots, 2p-1, \dots pq-1\}$ $\quad g_1 \cdot g_2 := g_1 \cdot g_2 \pmod N$

◆ order $(p-1)(q-1)$ ◆ not cyclic

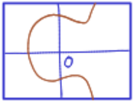
Hard in its cyclic subgroup

Elliptic curves modulo prime p

$(E, +)$ "curve points addition"

solutions to

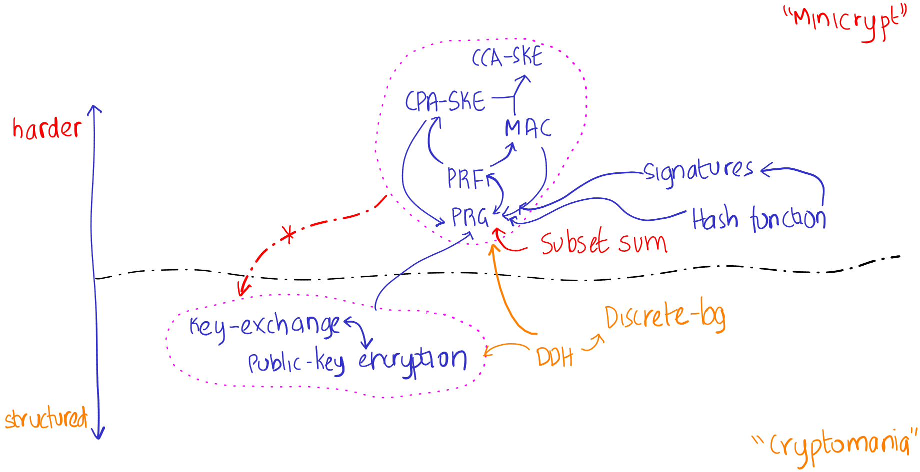
$y^2 = x^3 + Ax + B \pmod p$



◆ $|p+1 - \text{order}| \leq 2\sqrt{p}$ ◆ cyclic

Believed hard

What Else Can be Built from DDH?



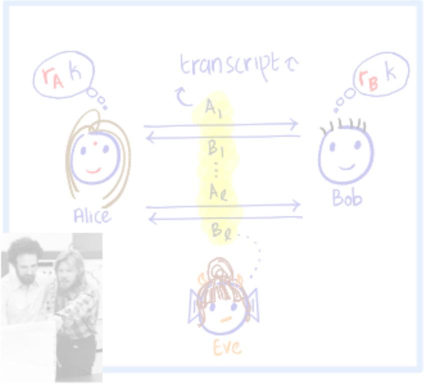
Exercise 2

Construct a PRG from DDH

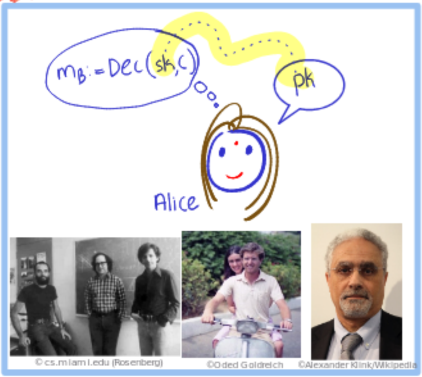
Plan for Today's Lecture

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange

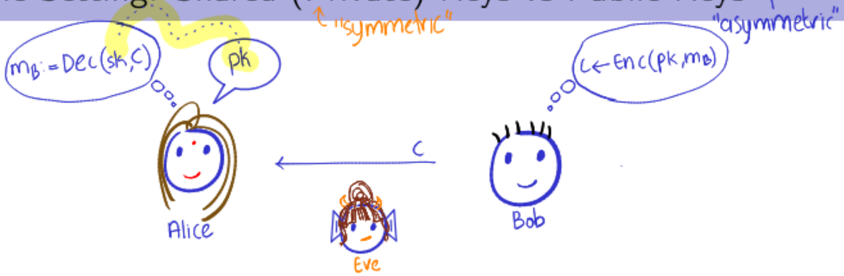


NEW Public-Key Encryption



Underlying hard problem: Decisional Diffie-Hellman (DDH)

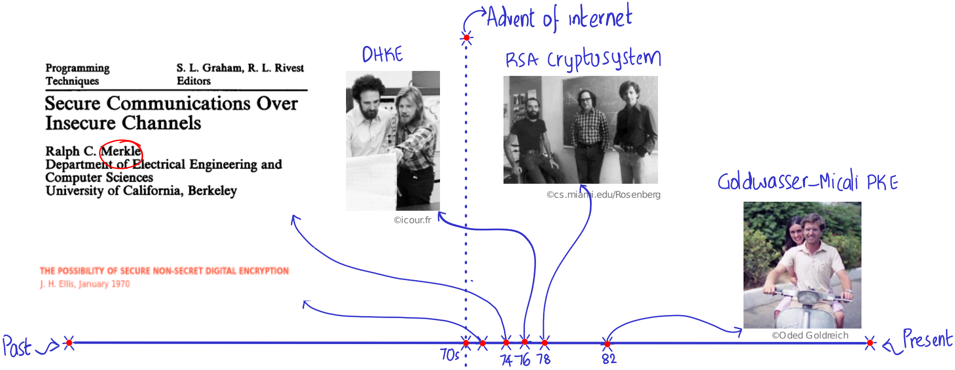
The Setting: Shared (Private) Keys vs Public Keys



- Recall the SKE setting: Alice and Bob *share* $k \in \{0,1\}^n$ and want to securely communicate in presence of *eavesdropper* Eve
- The *public-key* setting:
  1. Alice announces a *public key pk*; known to *everyone*!
  2. Bob wants to use *pk* to secretly send a message *to* Alice in presence of Eve
  3. Alice decrypts using her *secret key* *sk* (related to *pk*)

+ Advantage: scalability! It suffices to have one "key" per user

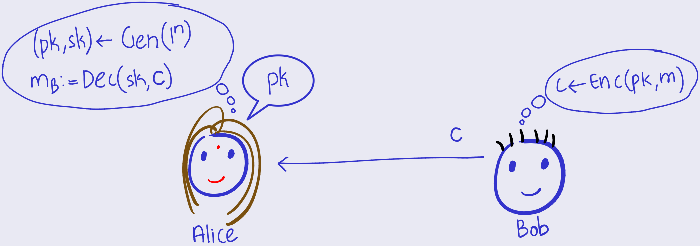
The Setting: Shared (Private) Keys vs Public Keys...



- PKE IRL: PGP, hybrid encryption

Definition 4 (Public-Key Encryption (PKE))

A PKE Π is a triple of efficient algorithms $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ with the following syntax:



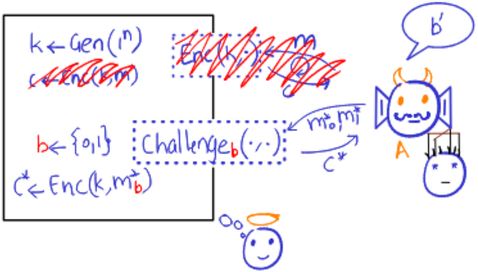
- Correctness of decryption: for every $n \in \mathbb{N}$, message $m \in \mathcal{M}_n$,

$$\Pr_{(pk,sk)\leftarrow \mathsf{Gen}(1^n),\, c\leftarrow \mathsf{Enc}(pk,m)}[\mathsf{Dec}(sk,c) = m] = 1$$

How can an unbounded eavesdropper Eve break PKE?

How to Define Security?

- Recall CPA-secrecy requirement in the SKE setting
- What is different in the PKE setting?
  - The public key known to Eve ⇒ encryption oracle "redundant"



How to Define Security? ...

- Recall CPA-secrecy requirement in the SKE setting
- What is different in the PKE setting?
  - The public key known to Eve ⇒ encryption oracle "redundant"
  - Eavesdropper=chosen-plaintext attacker!

Definition 5 (CPA Secrecy for PKE)

A PKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is CPA-secret if for *every* PPT (stateful) eavesdropper *Eve*, the following is negligible:

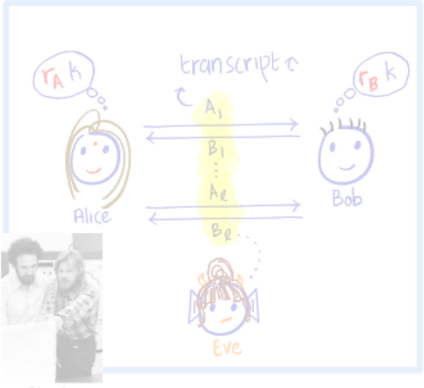
$$\delta(n) := \left| \Pr_{\substack{(pk,sk)\leftarrow\mathsf{Gen}(1^n)\\(m_0,m_1)\leftarrow\mathsf{Eve}(pk)\\c\leftarrow\mathsf{Enc}(pk,m_0)}}[\mathsf{Eve}(c)=0] - \Pr_{\substack{(pk,sk)\leftarrow\mathsf{Gen}(1^n)\\(m_0,m_1)\leftarrow\mathsf{Eve}(pk)\\c\leftarrow\mathsf{Enc}(pk,m_1)}}[\mathsf{Eve}(c)=0] \right|$$

"Left world" "Right world"

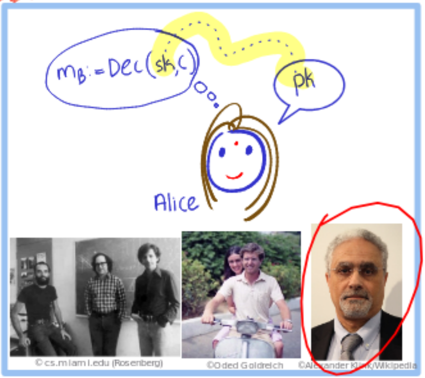
- Alternative, equivalent notion: semantic security
  - Ciphertext doesn't leak (non-trivial) information about plaintext
- Stronger notion: ind. against chosen-ciphertext attack (CCA)

- Task: public-key encryption
- Threat model: IND-CPA

NEW Diffie-Hellman Key Exchange

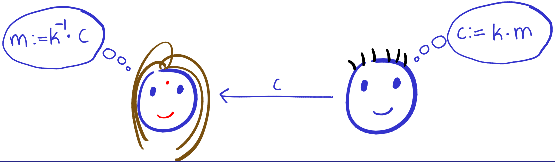


NEW **Public-Key Encryption**



Underlying hard problem: Decisional Diffie-Hellman (DDH)

One-Time Pad Can be Generalised Over Groups



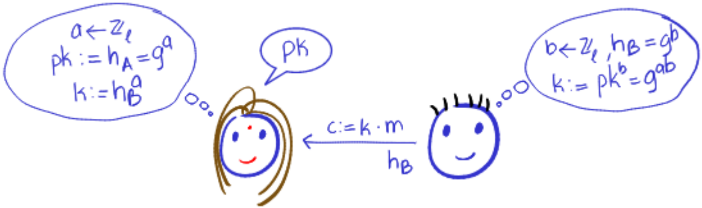
Pseudocode 1 (OTP over $(\{0,1\}^n, \oplus)$ with message space $\{0,1\}^n$)

- Key generation Gen: output $k \leftarrow \{0,1\}^n$
- Encryption Enc(k, m): output $c := k \oplus m$
- Decryption Dec(k, c): output $m := k \oplus c$

Pseudocode 2 (OTP over group $\mathbb{G} := (\mathcal{G}, \cdot)$ with message space $\mathcal{G}$)

- Key generation Gen: output $k \leftarrow \mathcal{G}$
- Encryption Enc(k, m): output $c := k \cdot m$
- Decryption Dec(k, c): output $m := k^{-1} \cdot c$

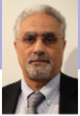
Let's Build on Group-Based OTP to Construct a PKE



- Our ciphertexts will be of form $c := k \cdot m$
- We need:
  1. Structure: two ways to generate the OTP k
  2. Eve mustn't be able to generate this k from pk and ciphertext c

Any ideas on
  1. What can the public key pk be?
  2. How to generate k?

Hint: we have already exploited this "structure" in DHKE



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ElGamal PKE over Group $\mathbb{G}$...

Pseudocode 3 (ElGamal PKE over group $\mathbb{G} = (\mathcal{G}, \cdot)$)

- Key generation $\mathsf{Gen}(1^n)$:
  1. Sample group $(\mathbb{G}, \ell, g) \leftarrow \mathsf{S}(1^n)$
  2. Sample random index $a \leftarrow \mathbb{Z}_\ell$
  3. Output $(pk := g^a, sk := a)$
- Encryption $\mathsf{Enc}(pk, m)$:
  1. Sample random index $b \leftarrow \mathbb{Z}_\ell$, and set $k := pk^b = (g^a)^b = g^{ab}$
  2. Output $c := (c_1, c_2) := (k \cdot m, g^b)$
- Decryption $\mathsf{Dec}(sk, c =: (c_1, c_2))$: output $m := (c_2^{sk})^{-1} \cdot c_1$

  $(g^b)^a = \cancel{g^{ab}}$ $\qquad$ $\cancel{g^{ab}} \cdot m$

- Correctness of decryption:

ElGamal PKE is CPA-secret

Theorem 2 (DDH $\rightarrow$ CPA-PKE)

ElGamal PKE is CPA-secret under DDH assumption in $\mathbb{G}$ w.r.to S.

Proof sketch. Hybrid argument.

Four hybrids H_0, H_0', H_1', H_1

Left world H_0

$(g^a, (g^b, g^{ab} \cdot m_0))$

Right world H_1

$(g^a, (g^b, g^{ab} \cdot m_1))$

DDH

DDH

Hybrid world H_0'

$(g^a, (g^b, g^r \cdot m_0))$

Hybrid world H_1'

$(g^a, (g^b, g^r \cdot m_1))$

OTP

? Why is H_0/H_1 indistinguishable from H_0'/H_1'? DDH assumption

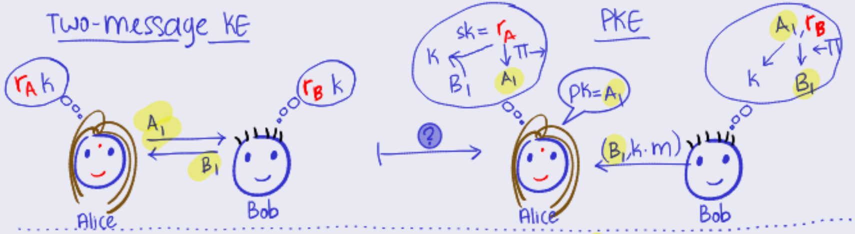
? Why is H_0' indistinguishable from H_1'? OTP over group

Is it a Coincidence that ElGamal is Similar to DHKE? No!

Claim 1 (Two-message KE → CPA-PKE)

If two-message key exchange protocol Π exists then so does PKE.

Construction 1



? How does Alice generate (pk, sk)? Generate first message A_1 using coins r_A

? How does Bob encrypt pt? Generate 2nd message B_1 and shared key k using coins r_B and A_1. Use k as OTP; send B_1 to help decryption.

Exercise 3 (Converse to Claim 1: two-message KE ← CPA-PKE)

If PKE exists then so does two-message key exchange.

Recap/Next Lecture

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- Diffie-Hellman key exchange (DHKE)
  - Based on DDH assumption in cyclic groups
  - Algebraic *structure* exploited: $(g^a)^b = g^{ab} = (g^b)^a$
- Public-key encryption (PKE)
  - Equivalent to two-round KE
  - Derived Elgamal PKE from DHKE

$a \leftarrow \mathbb{Z}_\ell$
$pk := h_A = g^a$
$k := h_B^a$

pk

$c := k \cdot m$

h_B

- Next lecture:
  - Factoring and related hardness assumptions
  - RSA group: multiplicative group modulo $N := pq$
  - Goldwasser-Micali encryption
  - RSA encryption

Key-exchange

Public-key encryption

DDH

Factoring

QR

References

1. [KL14, Chapter 11] for more details on key exchange
2. Read the seminal paper by Diffie and Hellman [DH76] for a description of the namesake key-exchange. In general this paper is a very insightful read.
3. Boneh's survey [Bon98] is an excellent source on the DDH problem.

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