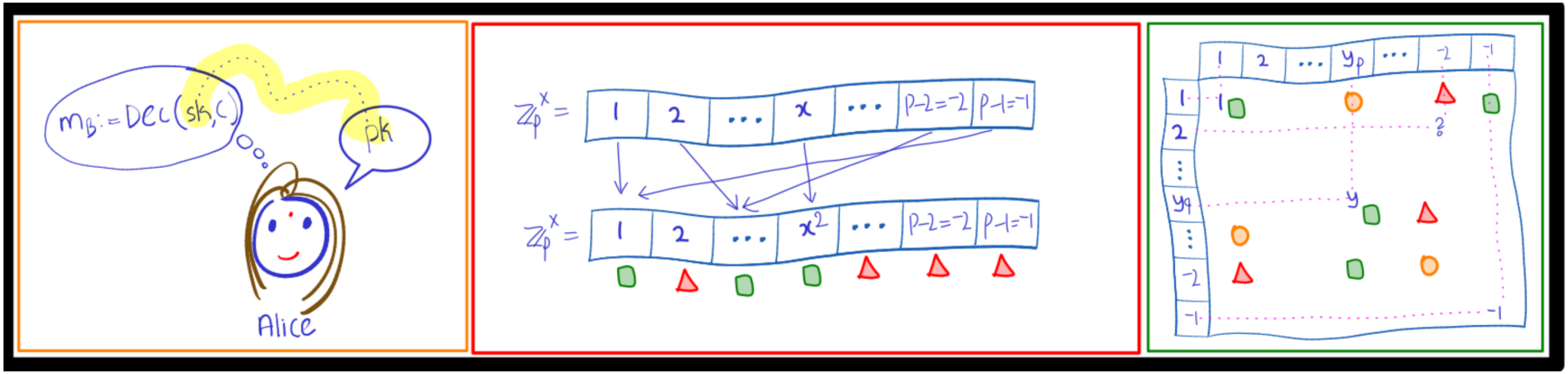
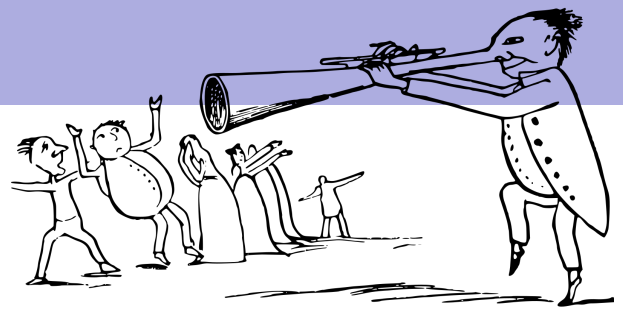


CS409m: Introduction to Cryptography

Lecture 14 (01/Oct/25)

Instructor: Chethan Kamath

Announcements



- Quiz 2 moved to 10/Oct (next Friday), 08:25-09:25, in CC103
- ⚠ Bounty on Problem 7.3 still on!
  - Come up with a *simple* construction of MAC from weak PRF
  - Construction provided in solution set is too complex!

Recall from Last Two Lectures...

- Tasks: key exchange (KEx) and public-key encryption (PKE)
- 2-Round KEx $\Leftrightarrow$ PKE

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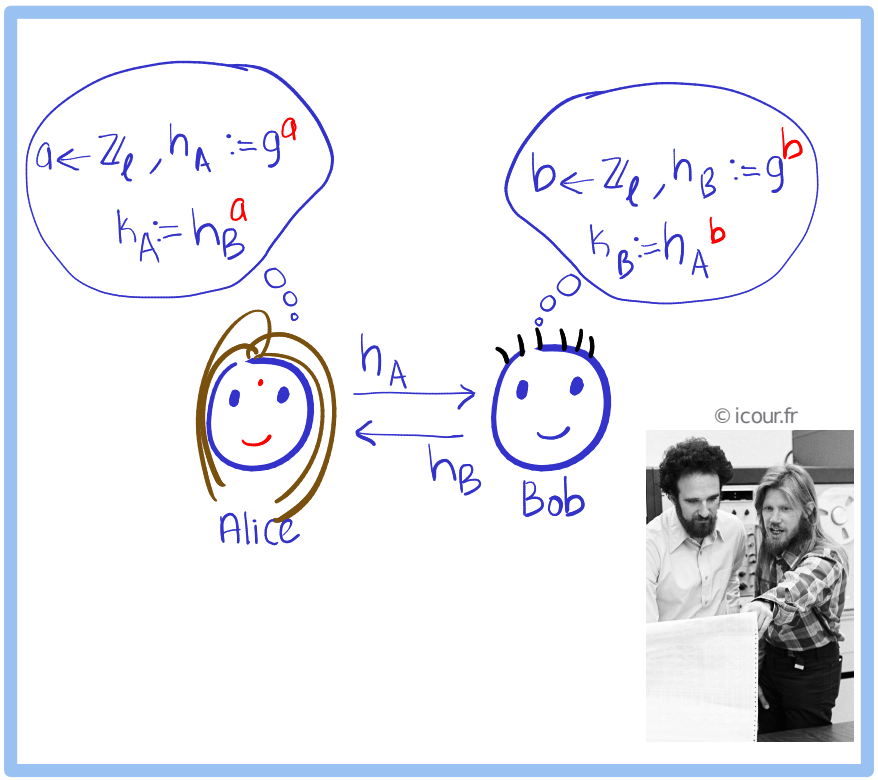
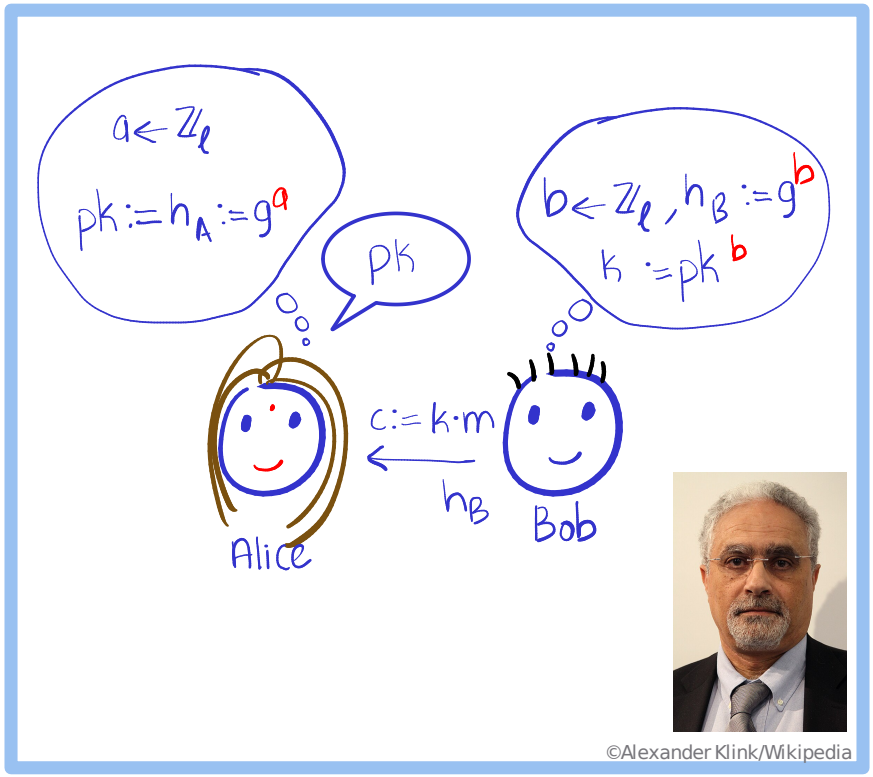
Diffie-Hellman KEx



Elgamal PKE



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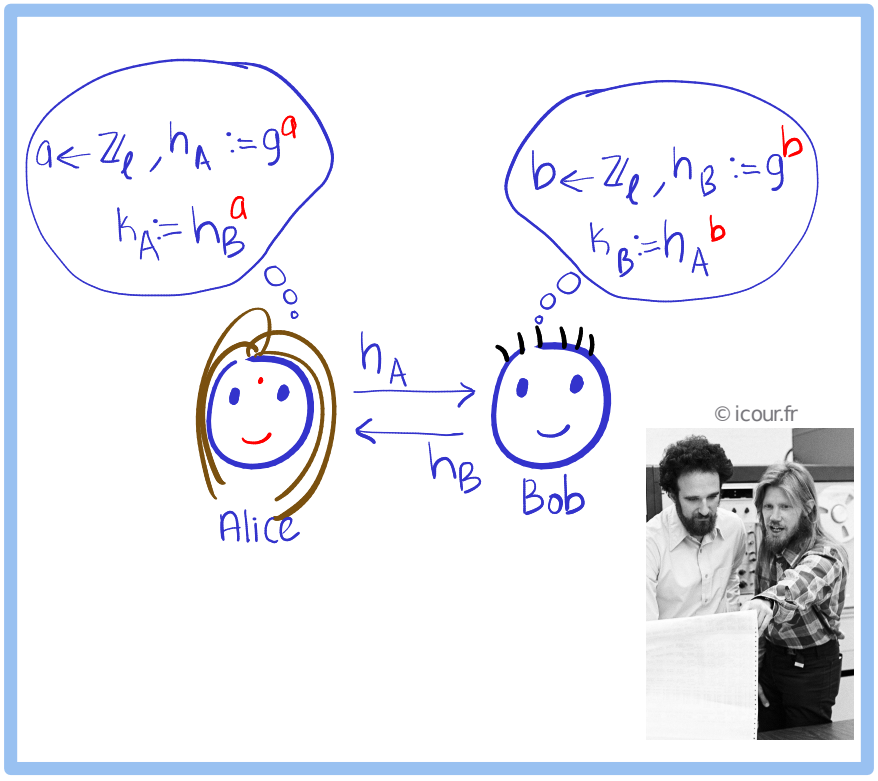
Diffie-Hellman KEx



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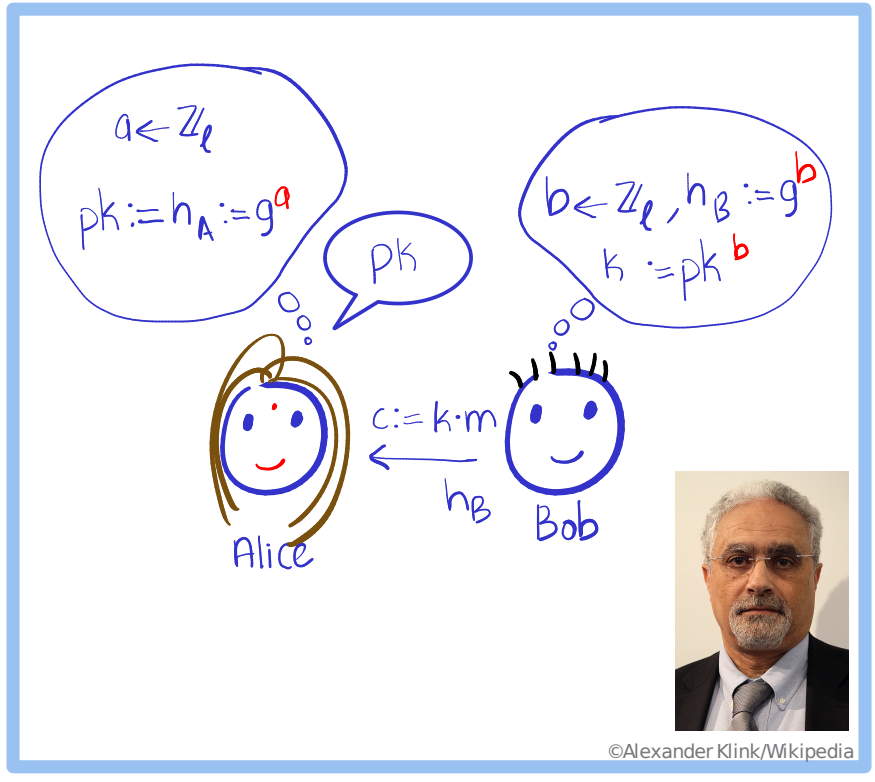
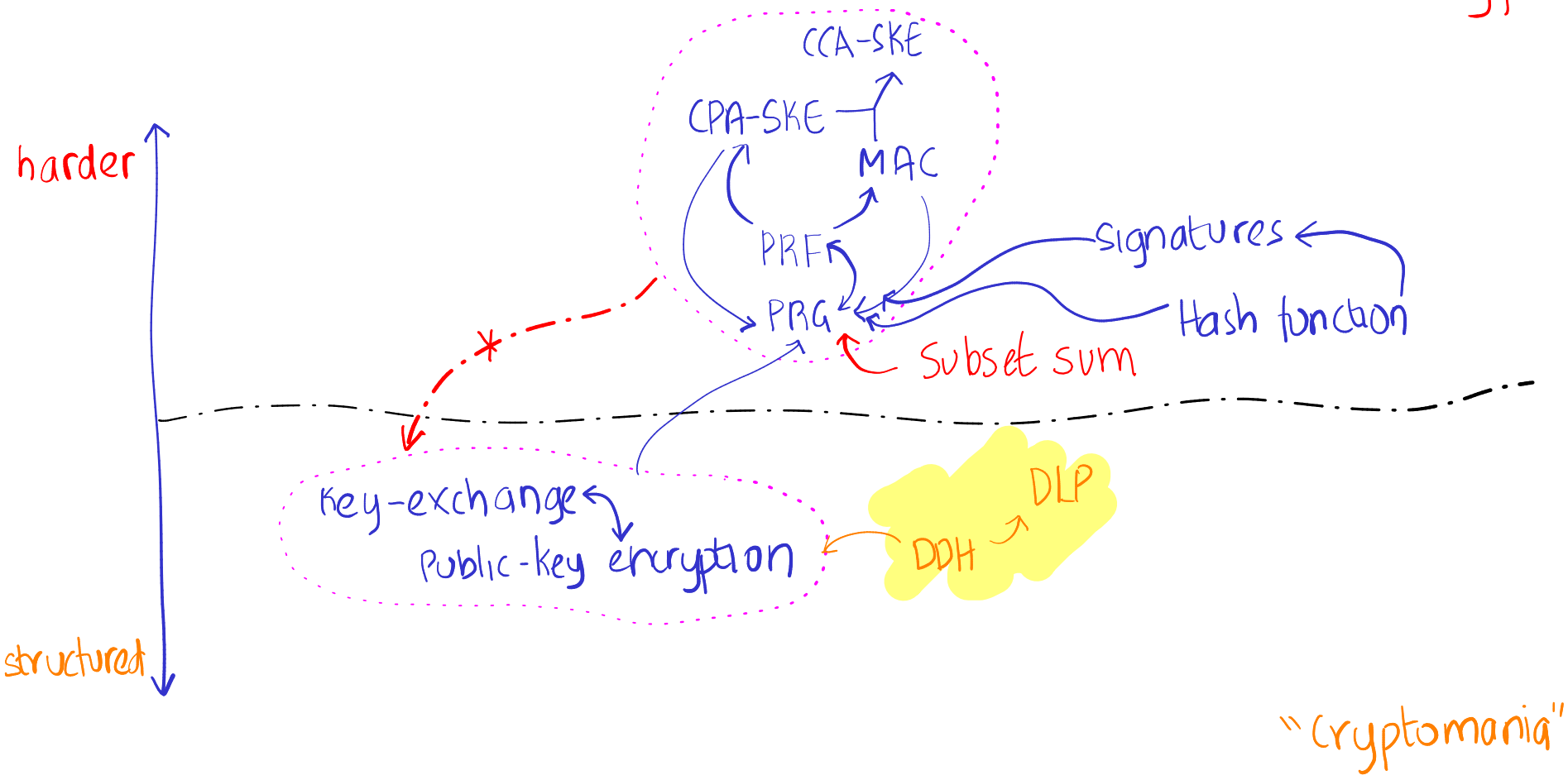


Structure we exploited: $(g^a)^b = g^{ab} = (g^b)^a$

Recall from Last Two Lectures...

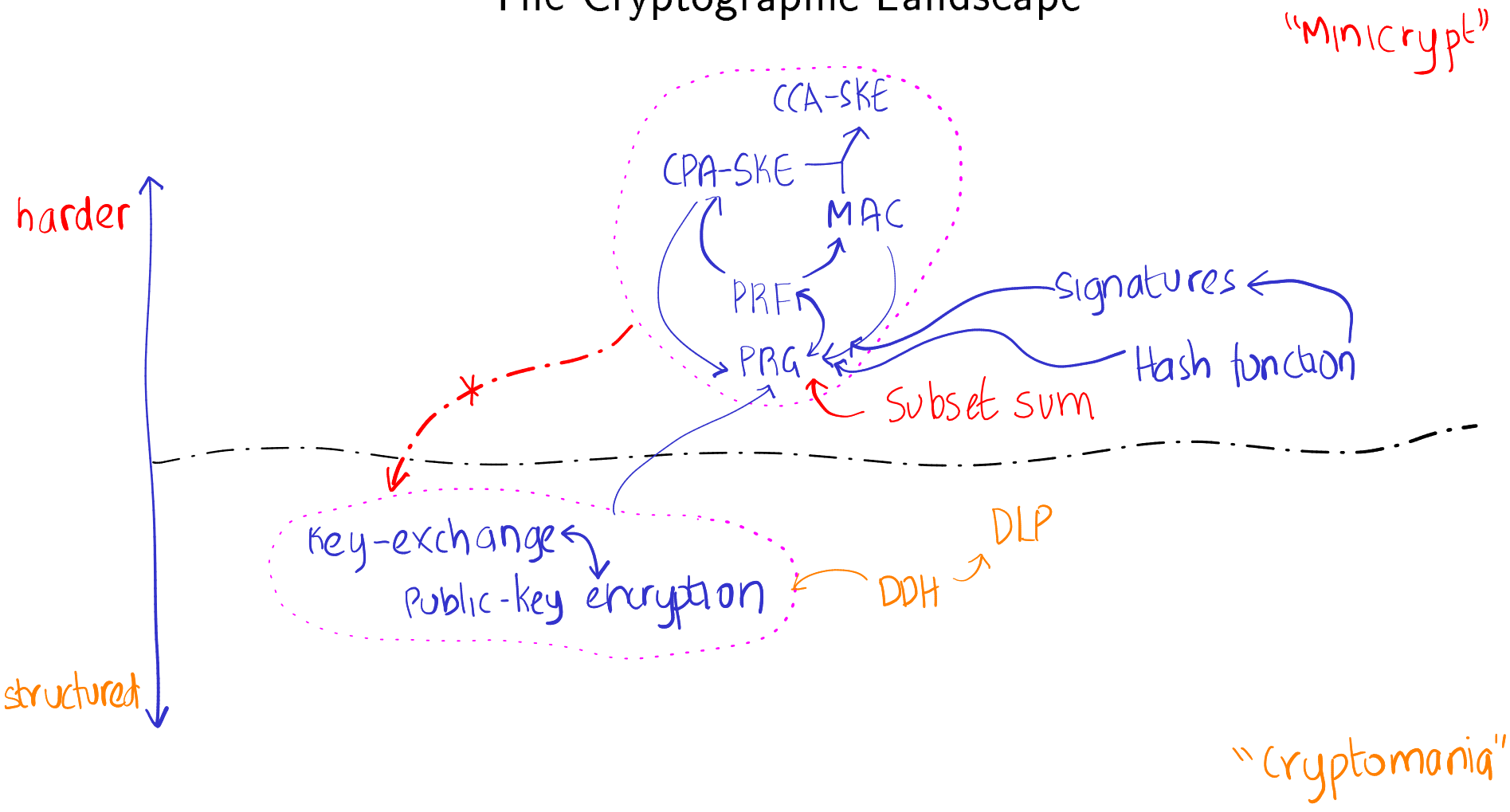
The Cryptographic Landscape



Group-based hard problems: DLP, CDH and DDH

Plan for Today's Lecture...

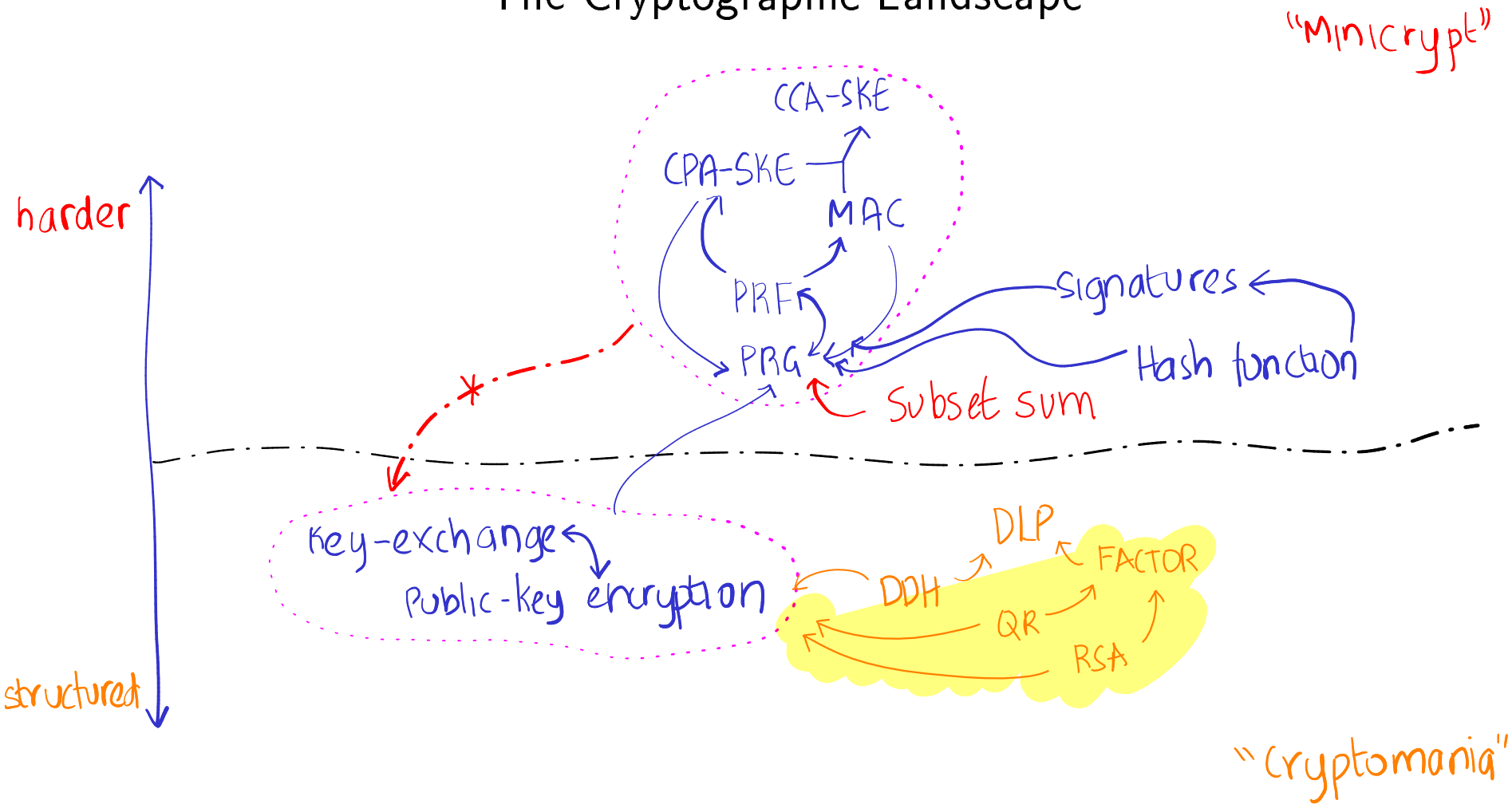
The Cryptographic Landscape



Today: Integer Factoring and its friends!

Plan for Today's Lecture...

The Cryptographic Landscape

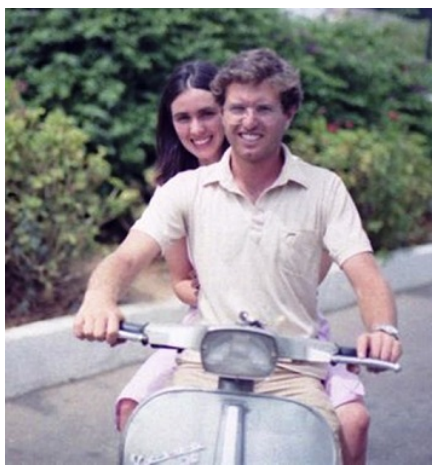


Today: Integer Factoring and its friends!

Plan for Today's Lecture...

- Task: public-key encryption (PKE)
- Threat model: IND-CPA

Goldwasser-Micali PKE



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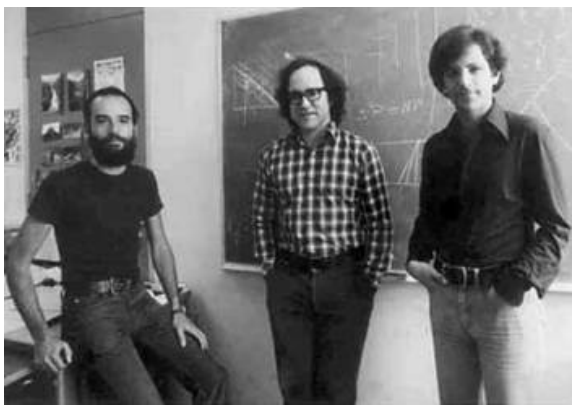
RSA PKE



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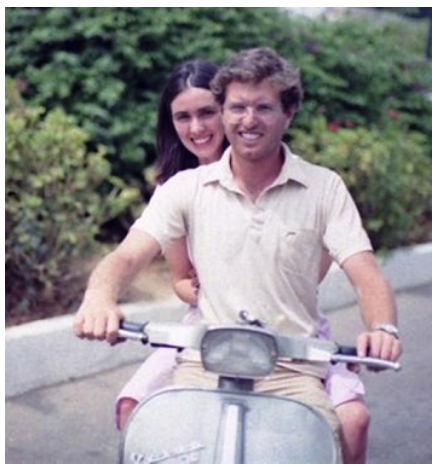

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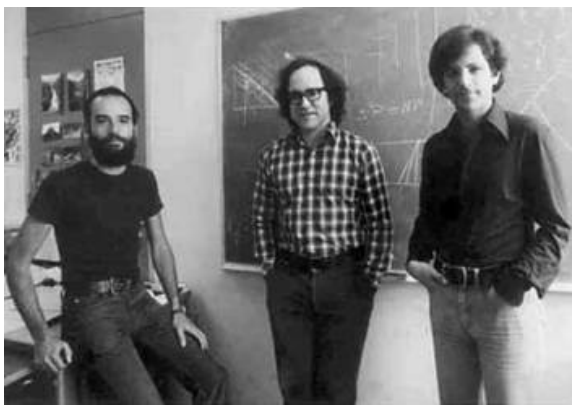
RSA PKE



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Algebraic setting: multiplication modulo *semiprime* (RSA group)

Plan for Today's Lecture

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Goldwasser-Micali PKE

RSA PKE

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 Algebraic setting: multiplication modulo *semiprime* (RSA group) 

Integer Factoring...

- Problem: Given a integer N, find $1 < p < N$ that divides N

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What about a *random* integer N?

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  - What seems hardest?

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  - What about a random *odd* integer N? N divisible by 3 with probability $1/3$ 
  - What seems hardest? *Semiprime*, i.e., N product of two primes

Integer Factoring...

Pseudocode 2

Semiprime sampler $S(1^n)$:

- Sample two random primes p and q of length $\approx n$
- Output $N := pq$

Integer Factoring...

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Assumption 1 (Factoring assumption w.r.to S . . .)

. . . holds if for all PPT A, the following is negligible:

$$\left|\delta(n) := \Pr_{N \leftarrow \mathsf{S}(1^n)}[\mathsf{A}(N) \text{ divides } N]\right|$$

Integer Factoring...

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Best known algorithm (Number-Field Sieve) requires $\approx 2^{|N|^{\frac{1}{3}}}$ time


Assumption does not hold against *quantum* adversaries!

- Shor's algorithm factors in *polynomial time* on quantum computer

Integer Factoring in the Wild!

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RSA Factoring Challenge

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RSA617	617	2048			
RSA2048	617	2048	US$200,000[d]		

Plan for Today's Lecture

- Task: public-key encryption (PKE)
- Threat model: IND-CPA

Goldwasser-Micali PKE

RSA PKE

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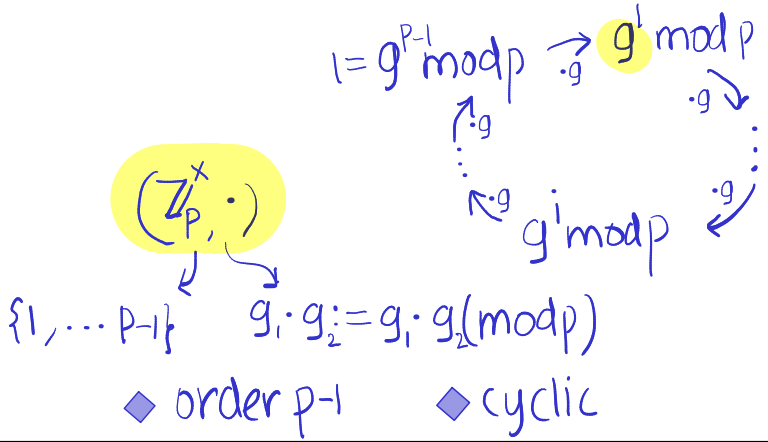
Algebraic setting: multiplication modulo *semiprime* (RSA group)

Multiplication Modulo Semiprime $N = pq$

Multiplication mod prime p



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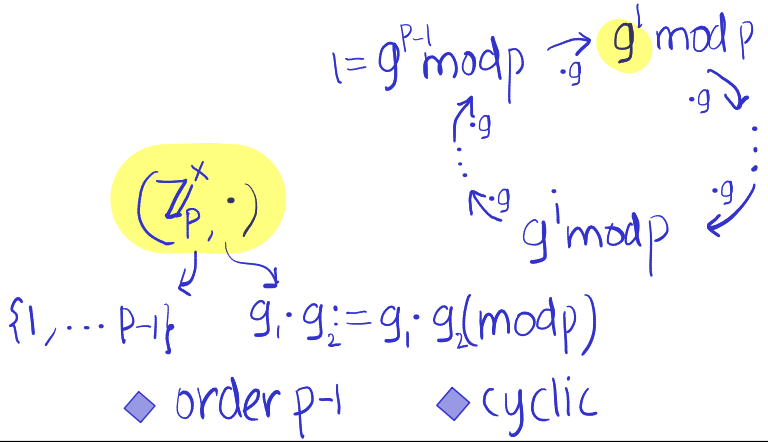
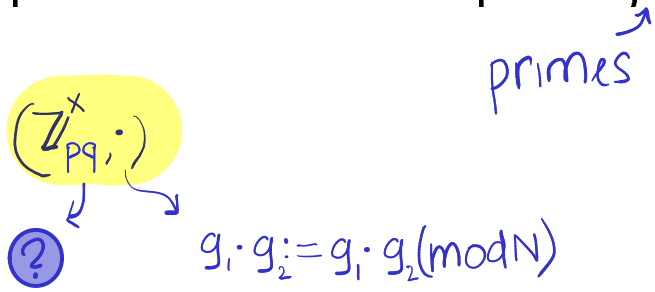


- order $p-1$
- cyclic

Multiplication mod semiprime pq



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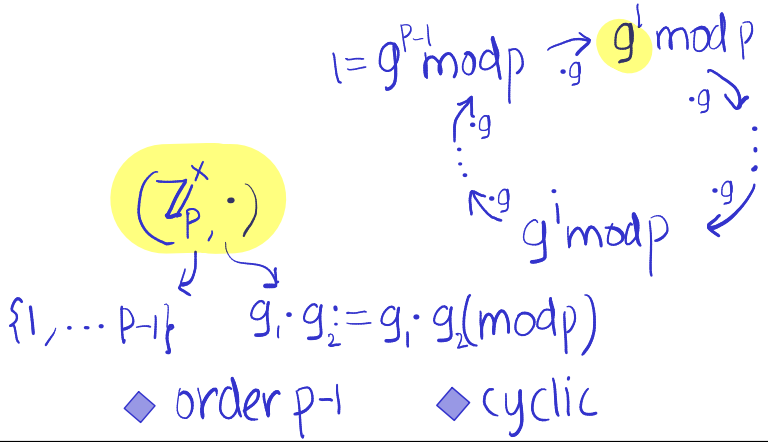

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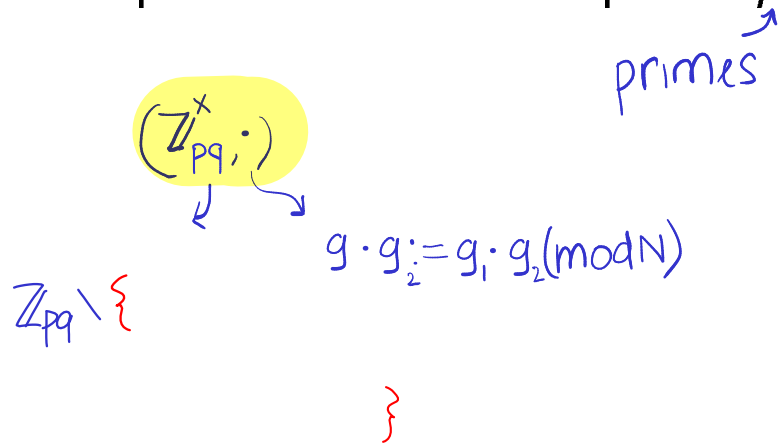

? What are the elements in $\mathbb{Z}_{pq}^\times$? Every $0 \leq a < N$ that is invertible

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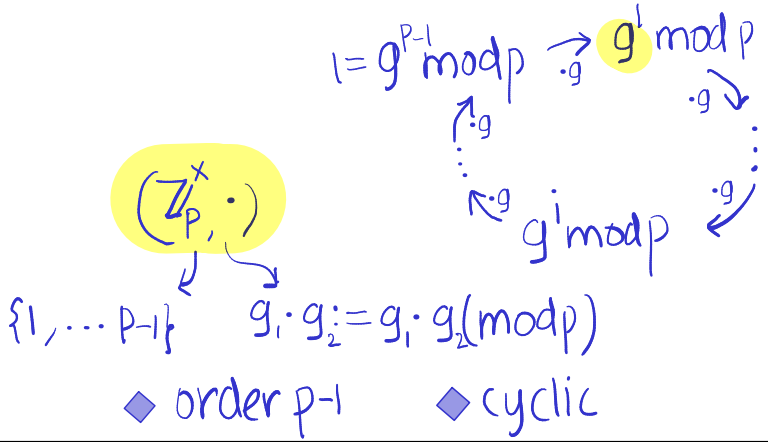
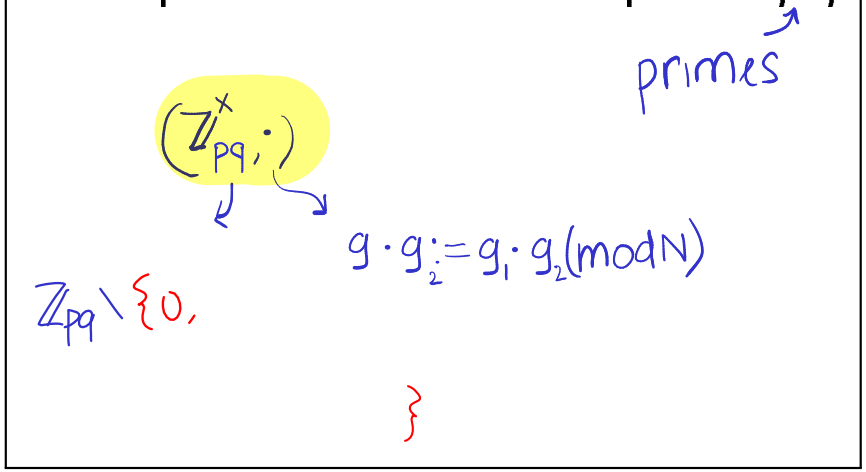


Multiplication mod semiprime pq



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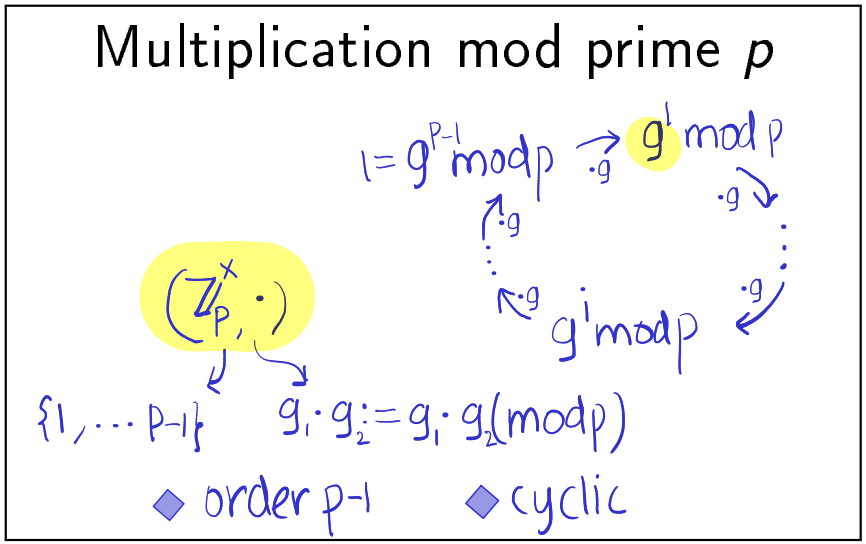

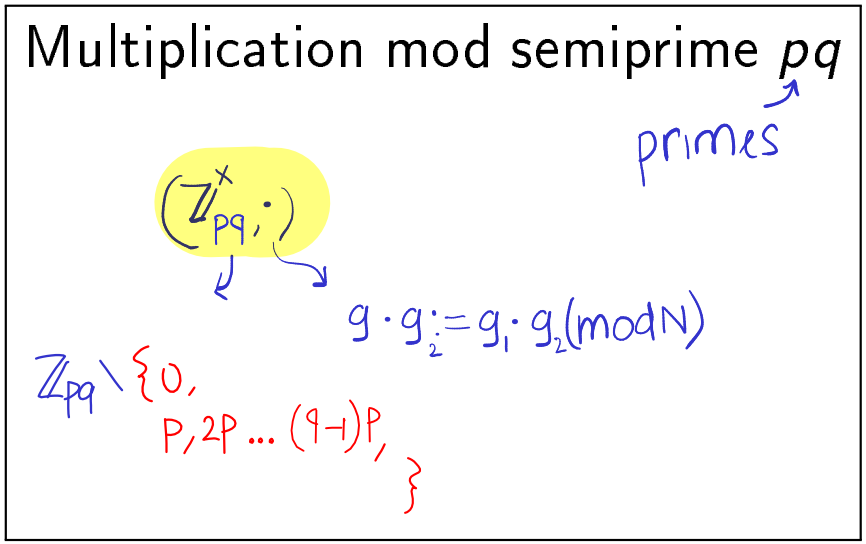

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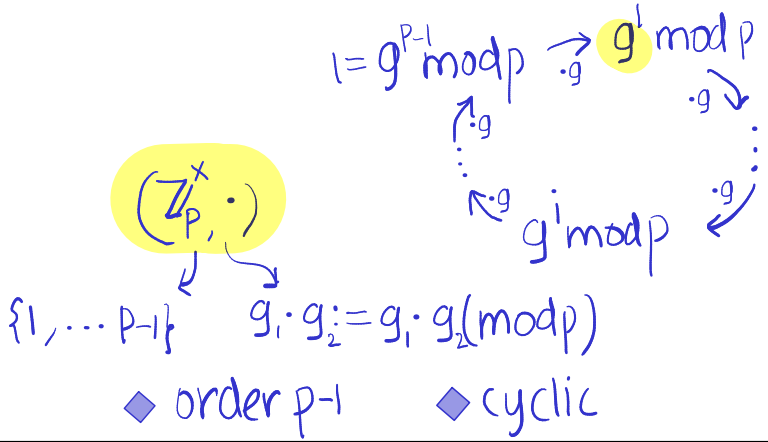
  - p and its multiples are not invertible (proof on whiteboard)

Multiplication Modulo Semiprime $N = pq$

Multiplication mod prime p



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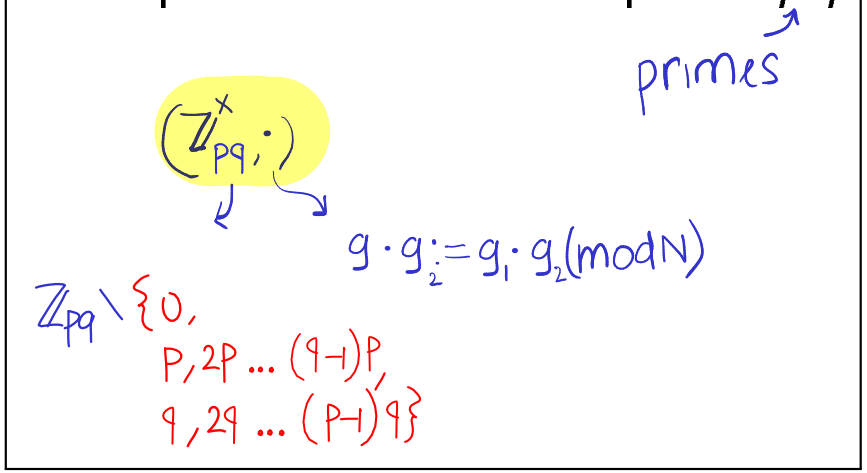

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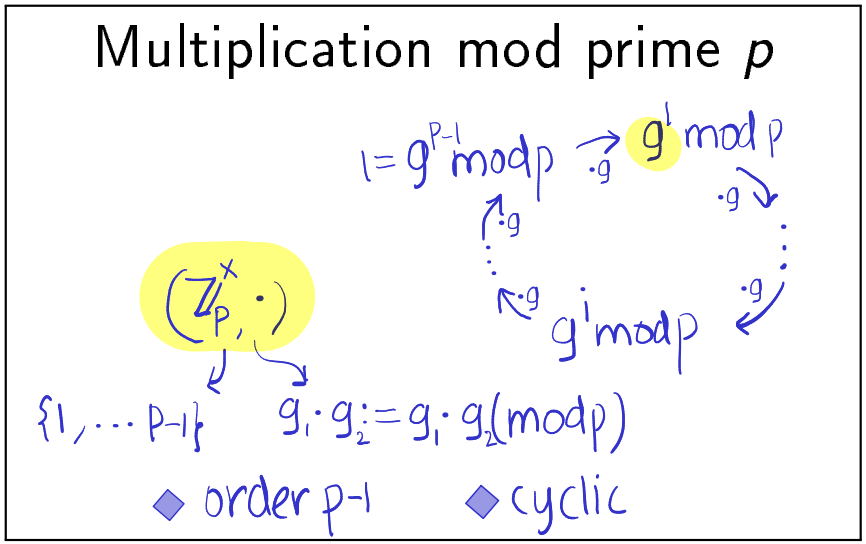
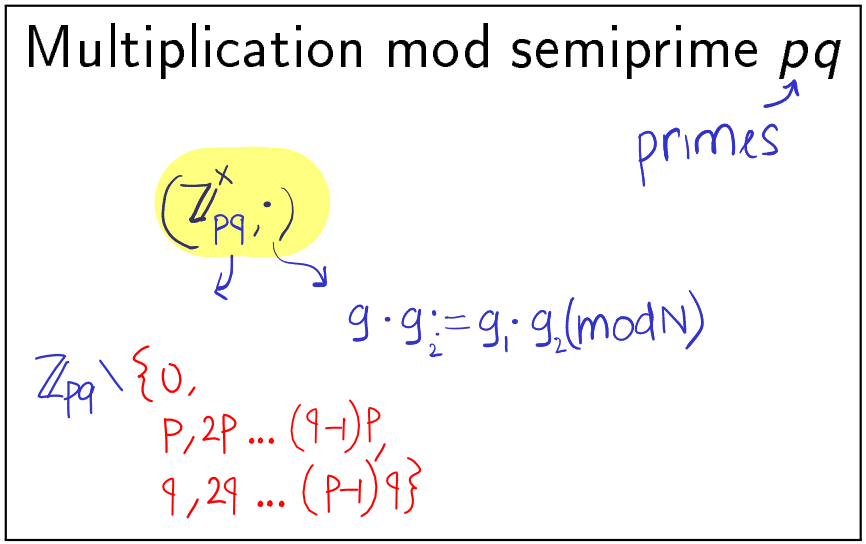
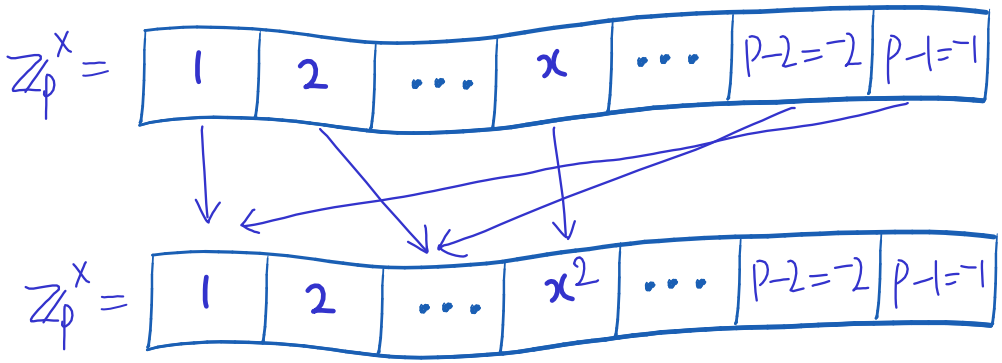




- What are the elements in $\mathbb{Z}_{pq}^\times$? Every $0 \leq a < N$ that is invertible
  - 0 is not invertible
  - p and its multiples are not invertible (proof on whiteboard)
  - q and its multiples are not invertible
- What is the order of the $\mathbb{Z}_{pq}^\times$?

$$\begin{aligned} pq - 1 - (q-1) - (p-1) &= pq - p - q - 1 \\ &= (p-1)(q-1) =: \phi(N) \end{aligned}$$

Squaring Map $x \mapsto x^2 \bmod p$



- 2-1 map

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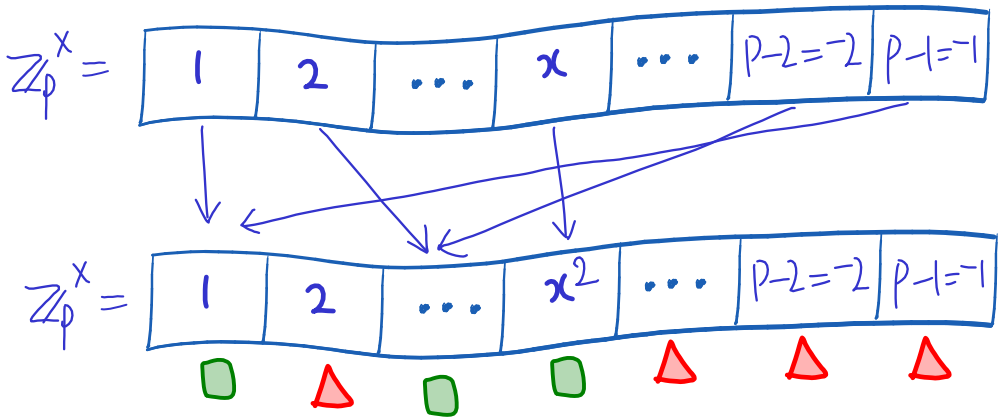
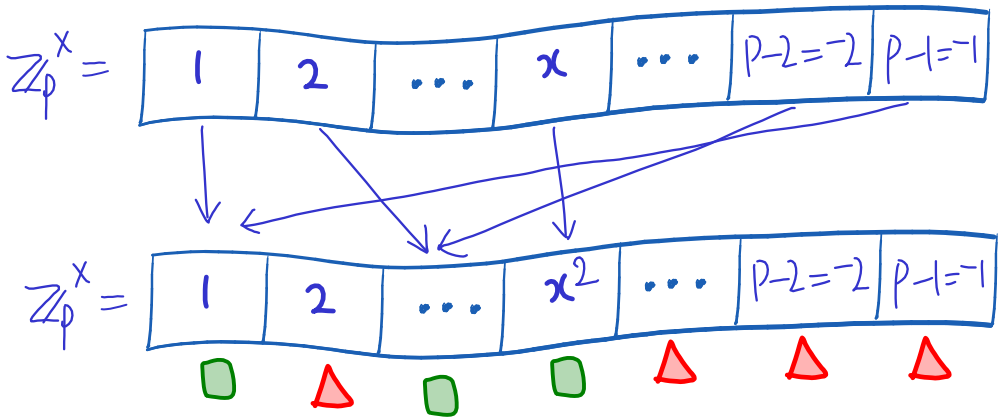


- 2-1 map $\Rightarrow$ Half the elements $\mathbb{Z}_p^\times(+) \subset \mathbb{Z}_p^\times$ have square roots
- Is it possible to *test* if $y \in \mathbb{Z}_p^\times(+)$?

Squaring Map $x \mapsto x^2 \bmod p$



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  1. Compute discrete log x of y w.r.to *some* generator g
  2. Output "square" if x is *even*

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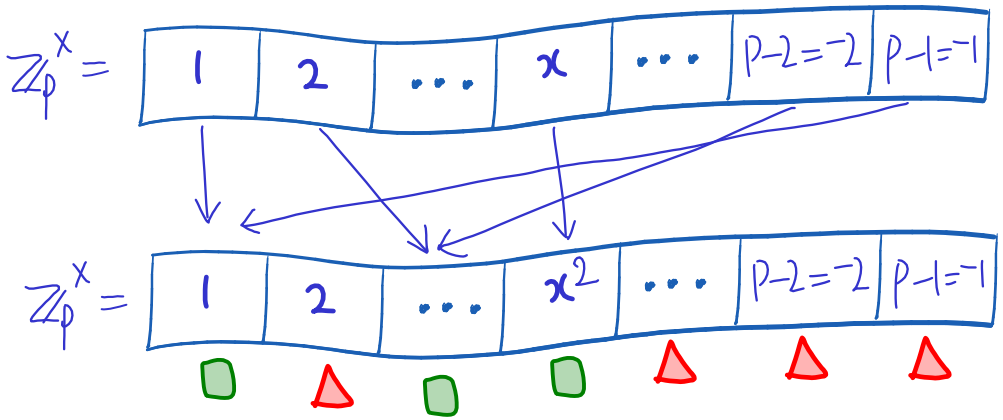

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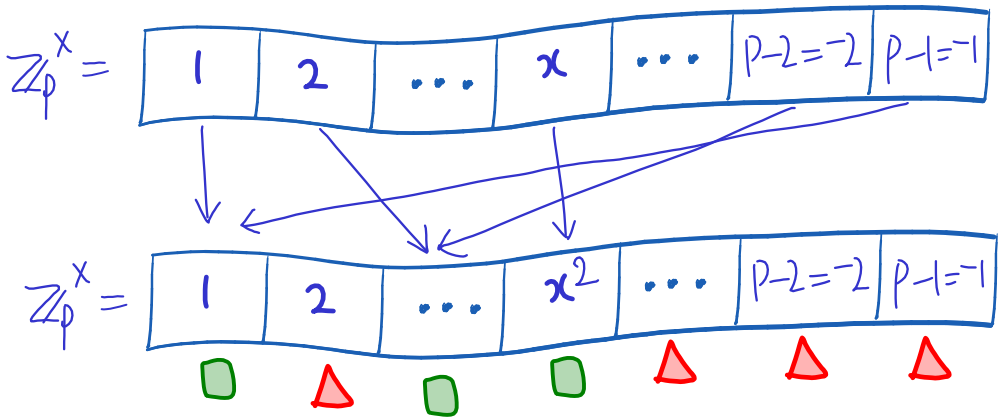

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- Is it possible to *efficiently* test if $y \in \mathbb{Z}_p^\times(+)$? Yes:
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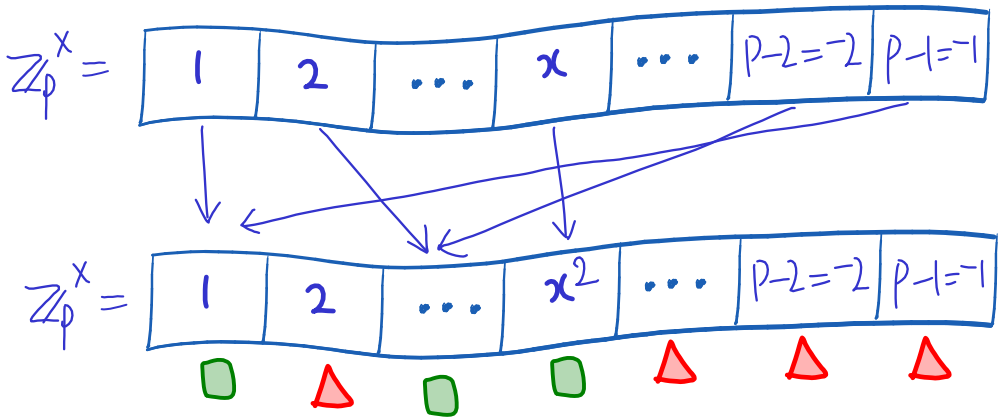
  2. Output "square" if sign is $+1$

Squaring Map $x \mapsto x^2 \bmod p$



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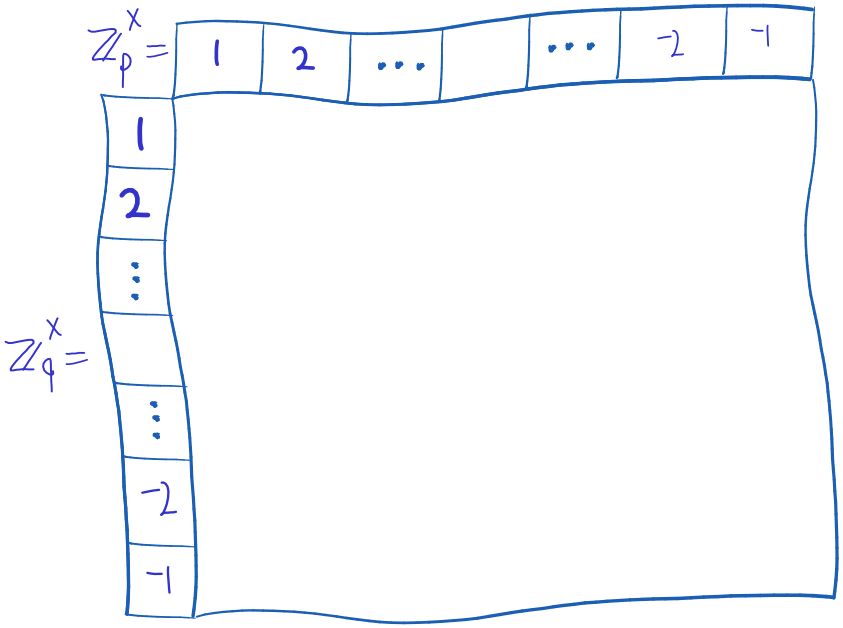
Exercise 1 (Exercise 2, Assignment 4)

Show that DDH assumption doesn't hold in $(\mathbb{Z}_p^\times, \cdot)$

Squaring Map $x \mapsto x^2 \bmod pq$...



- Chinese Remaindering Theorem: $\mathbb{Z}_N^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times$ (on whiteboard)

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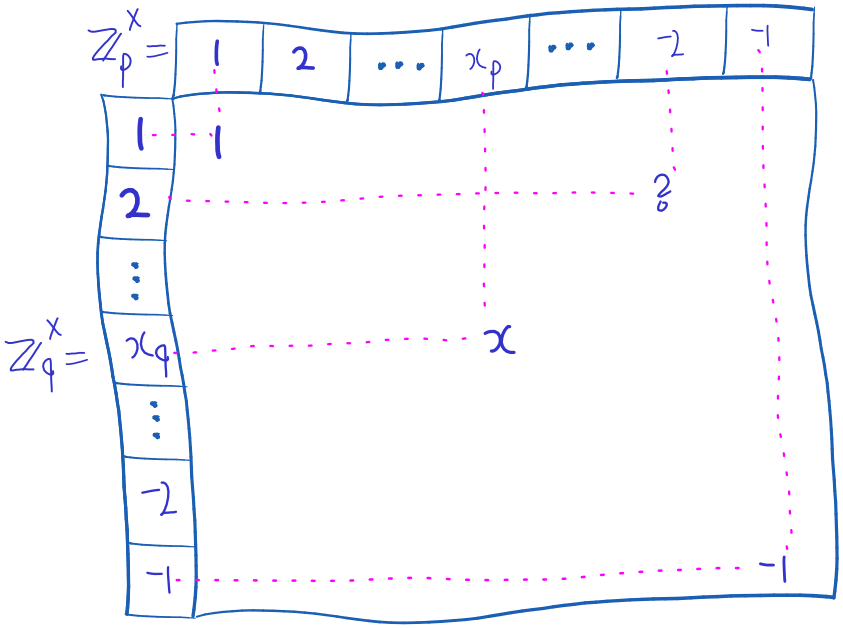

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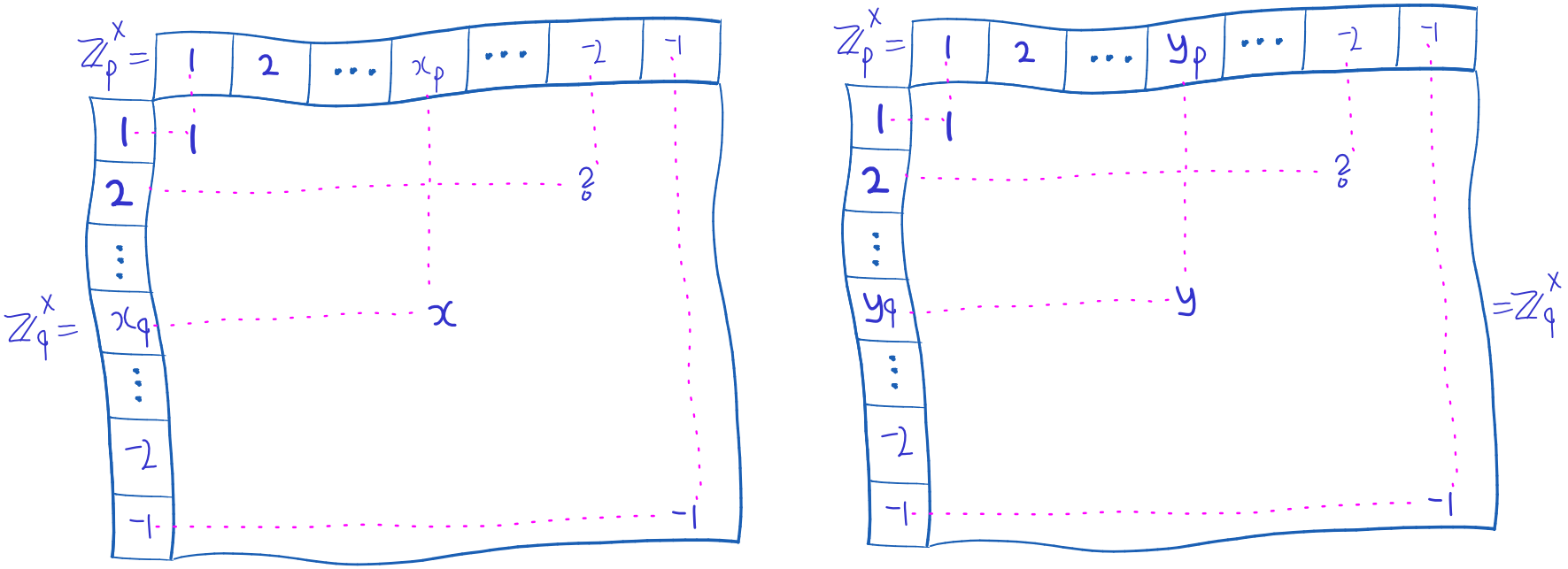

- Chinese Remaindering Theorem: $\mathbb{Z}_N^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times$ (on whiteboard)
- $\Rightarrow$ 4-1 map $\Rightarrow$ 1/4 of elements $\mathbb{Z}_N^\times(+,+) \subset \mathbb{Z}_N^\times$ have square roots

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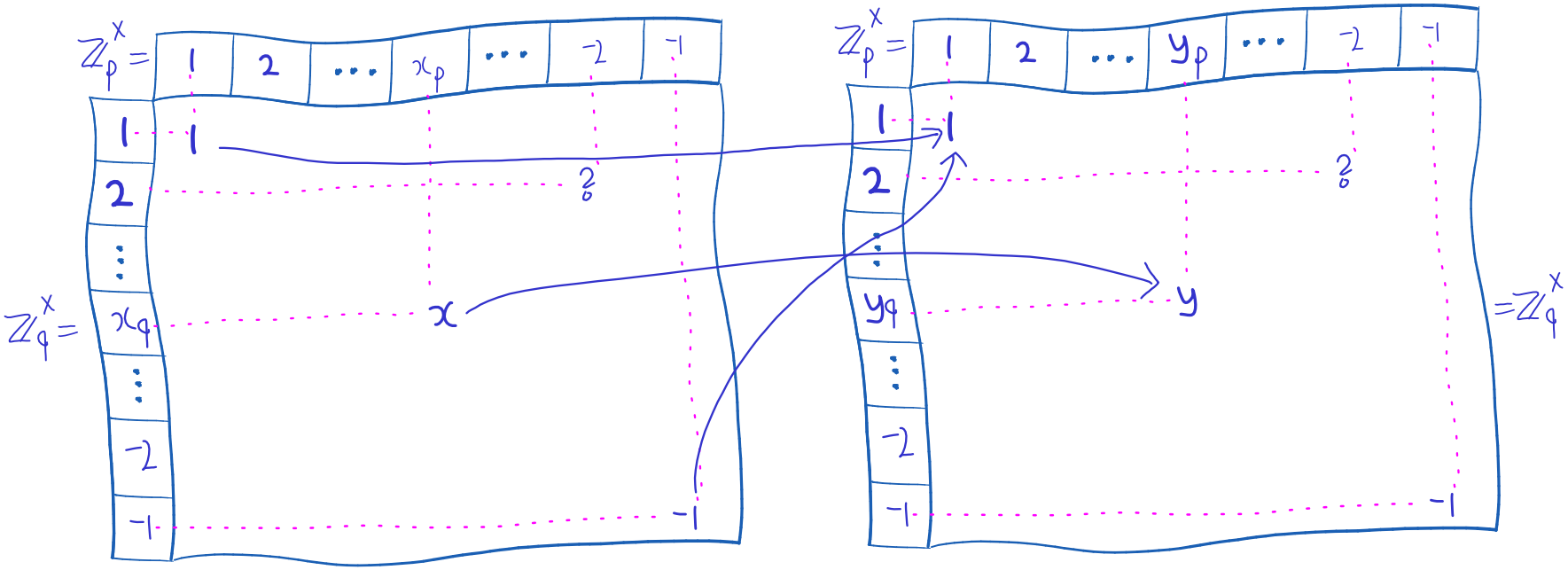

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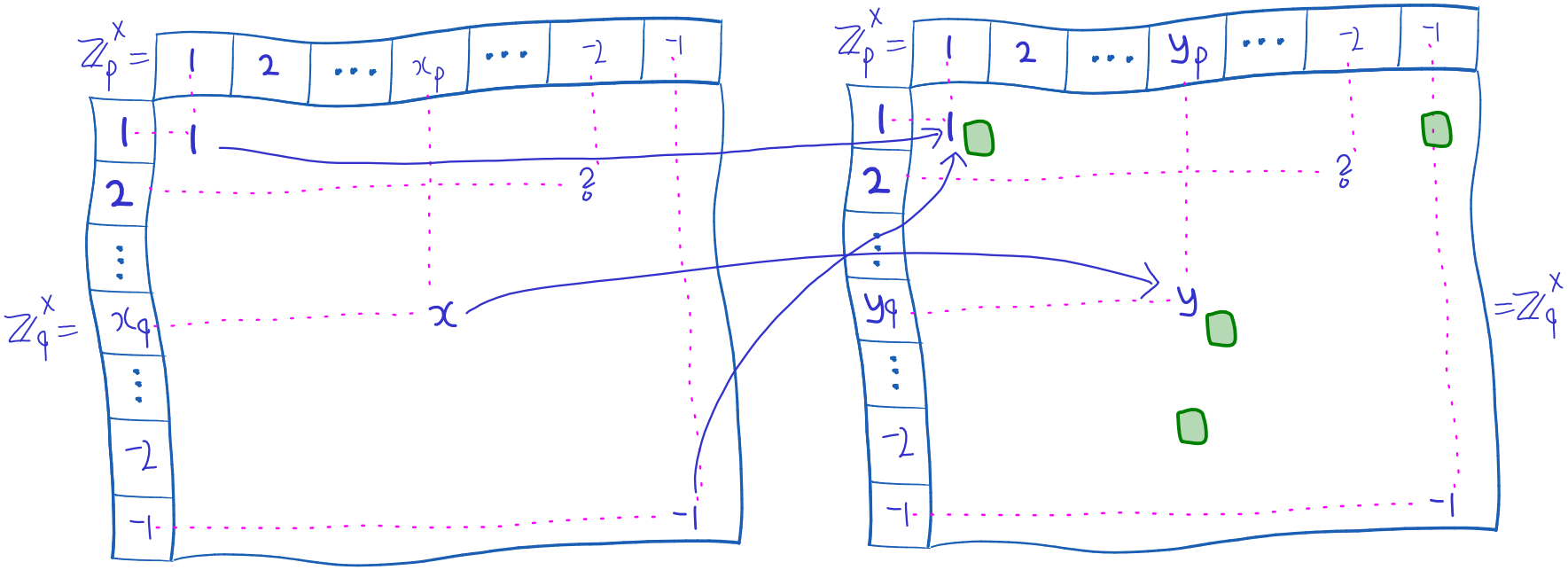

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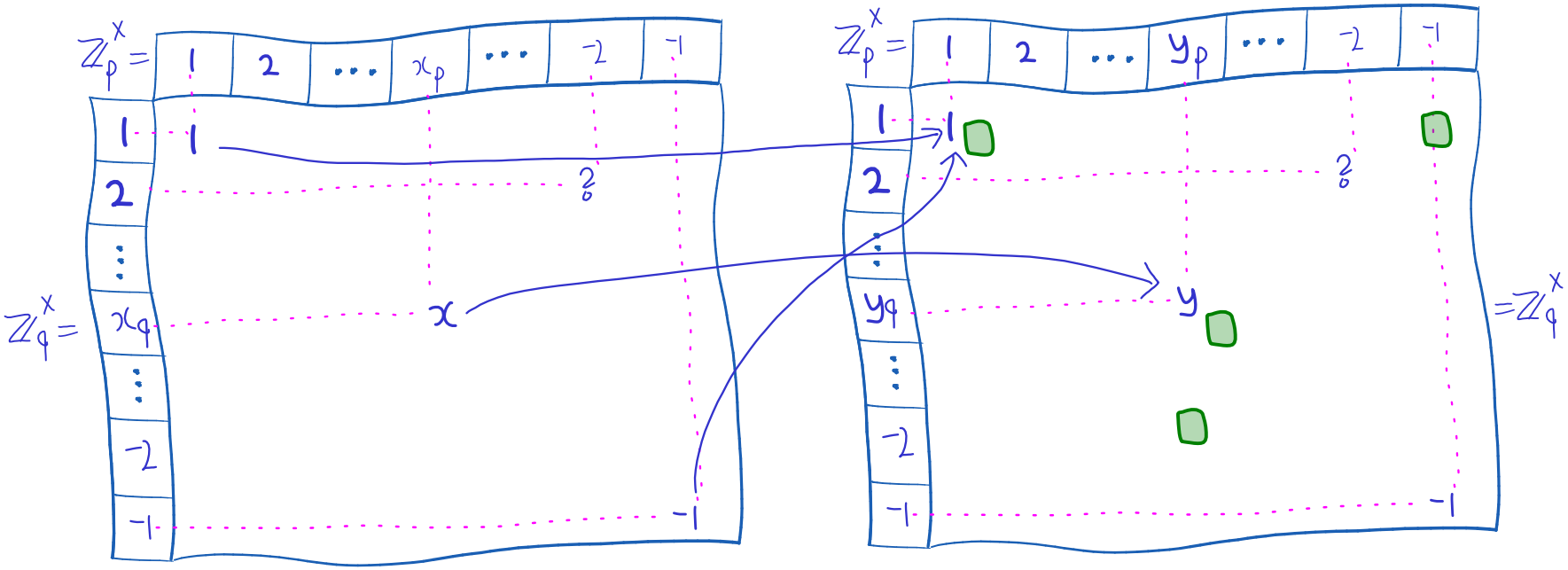

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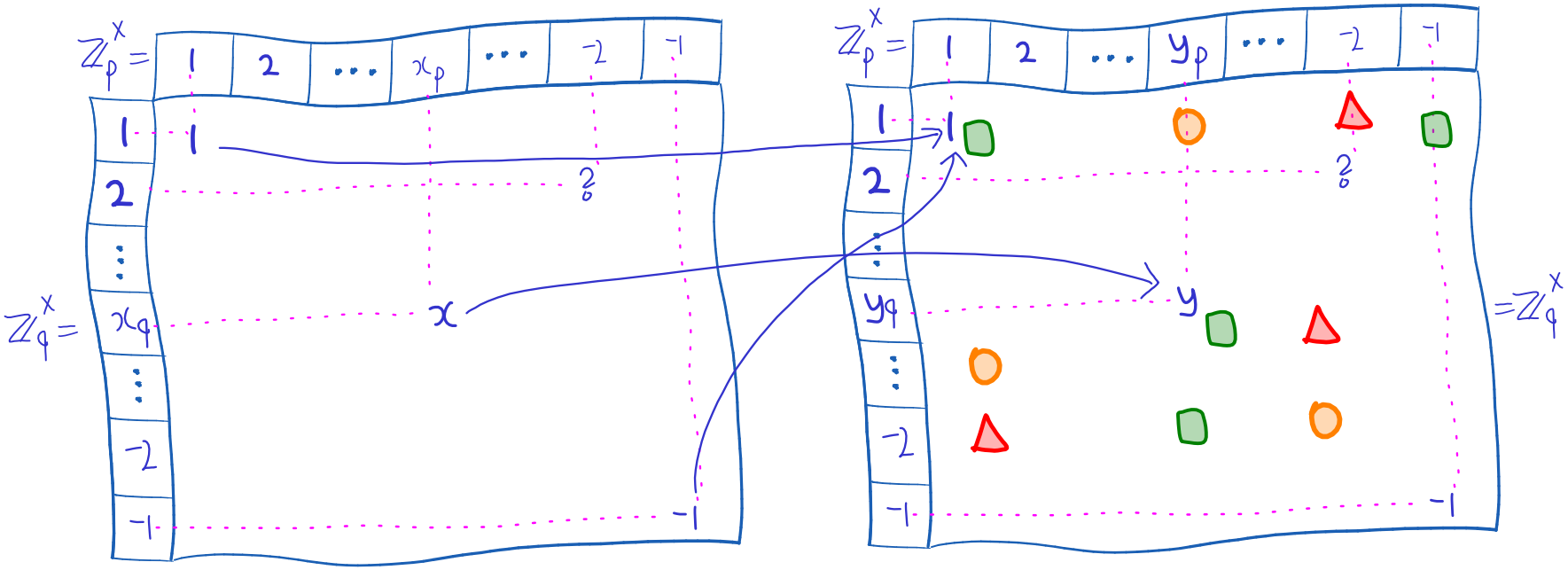

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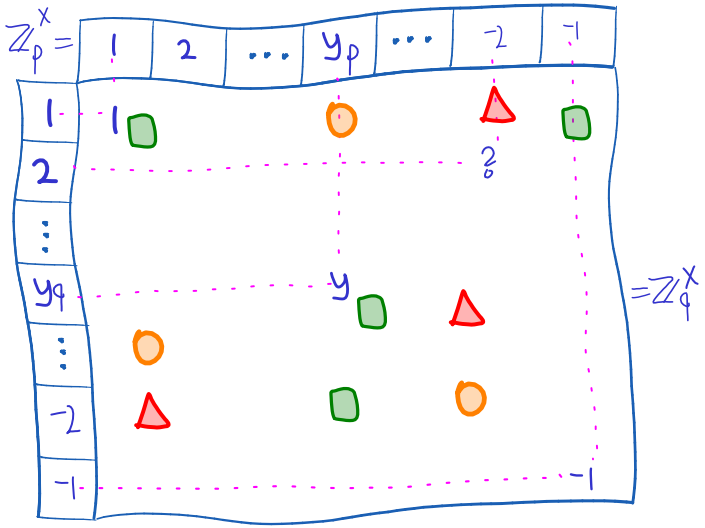
- Is it possible to *efficiently test* if $y \in \mathbb{Z}_N^\times(+,+)$? Unclear
- Can efficiently distinguish $\mathbb{Z}_N^\times(+,+) \cup \mathbb{Z}_N^\times(-,-)$ from $\mathbb{Z}_N^\times(-,+) \cup \mathbb{Z}_N^\times(+,-)$: compute Jacobi symbol

Squaring Map $x \mapsto x^2 \bmod pq$...



Assumption 2 (Quadratic residuosity (QR) assumption w.r.to S...)

...holds if for all PPT distinguishers D, the following is negligible:

$$\delta(n) := \left| \Pr_{\substack{N \leftarrow \mathsf{S}(1^n) \\ y \leftarrow \mathbb{Z}_N^\times(+,+)}} [\mathsf{D}(N, y) = 0] - \Pr_{\substack{N \leftarrow \mathsf{S}(1^n) \\ y \leftarrow \mathbb{Z}_N^\times(-,-)}} [\mathsf{D}(N, y) = 0] \right|$$

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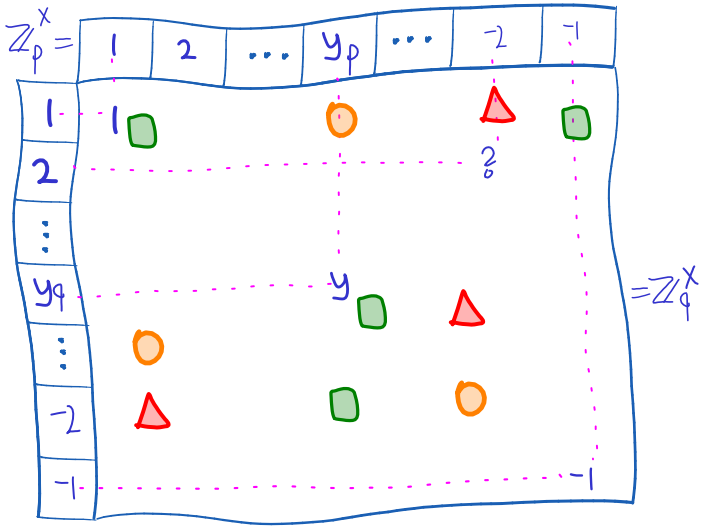

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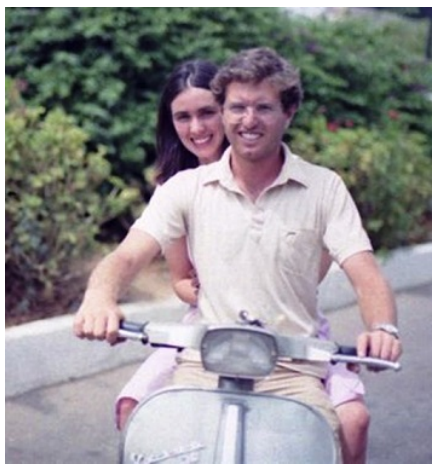
Exercise 2

1. Show that QR assumption implies Factoring assumption
2. Show that *computing* square root mod N is equivalent to factoring

Plan for Today's Lecture

- Task: public-key encryption (PKE)
- Threat model: IND-CPA

Goldwasser-Micali PKE



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RSA PKE

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Algebraic setting: multiplication modulo *semiprime* (RSA group)

Goldwasser-Micali Public-Key (Bit) Encryption

Key idea: encode message in the "sign"

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$$\begin{cases} 0 & \text{if } c \in \mathbb{Z}_N^\times(+,+) \cong \mathbb{Z}_p^\times(+) \times \mathbb{Z}_q^\times(+) \\ 1 & \text{otherwise} \end{cases}$$

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- Correctness of decryption: since $r^2 \in \mathbb{Z}_N^\times(+,+)$, $c \in \mathbb{Z}_N^\times(+,+)$ iff $m = 0$

Goldwasser-Micali PKE is CPA-secret

Theorem 1 (QR $\rightarrow$ IND-CPA security1)

Goldwasser-Micali PKE is CPA-secret under QR assumption.

Proof. $\exists$D against QR $\Leftarrow$ $\exists$Eve against PKE.

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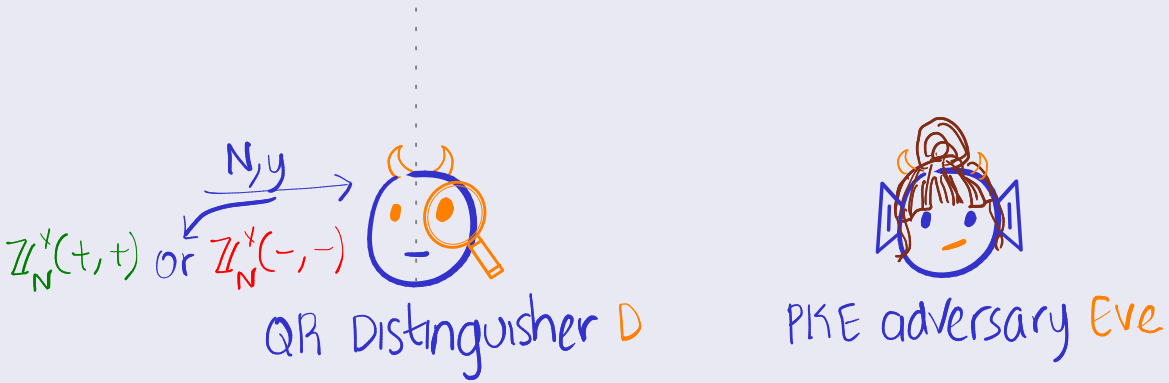

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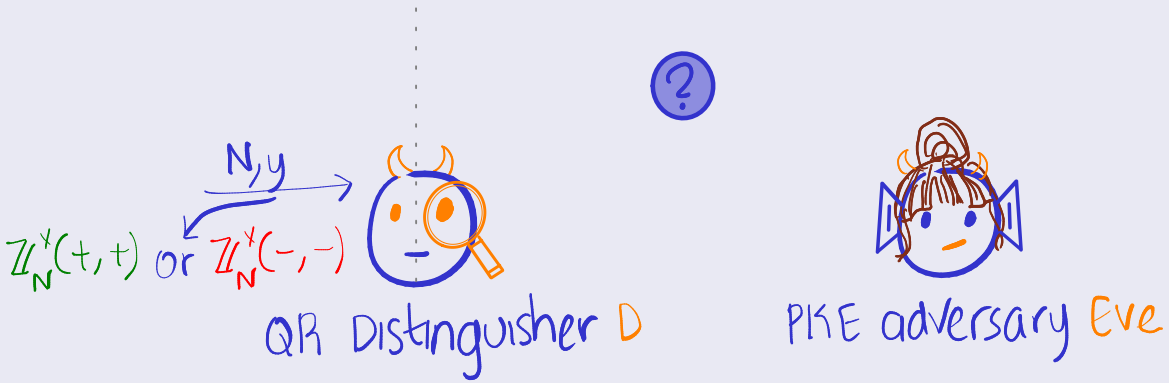

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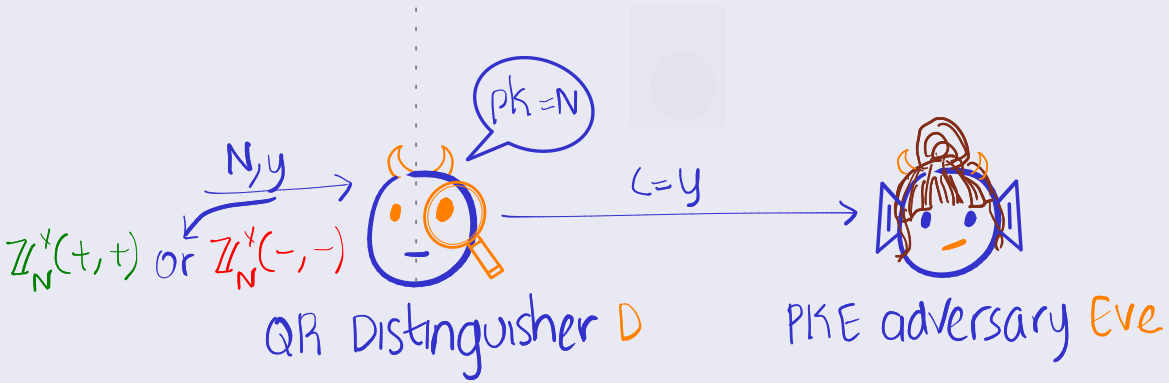

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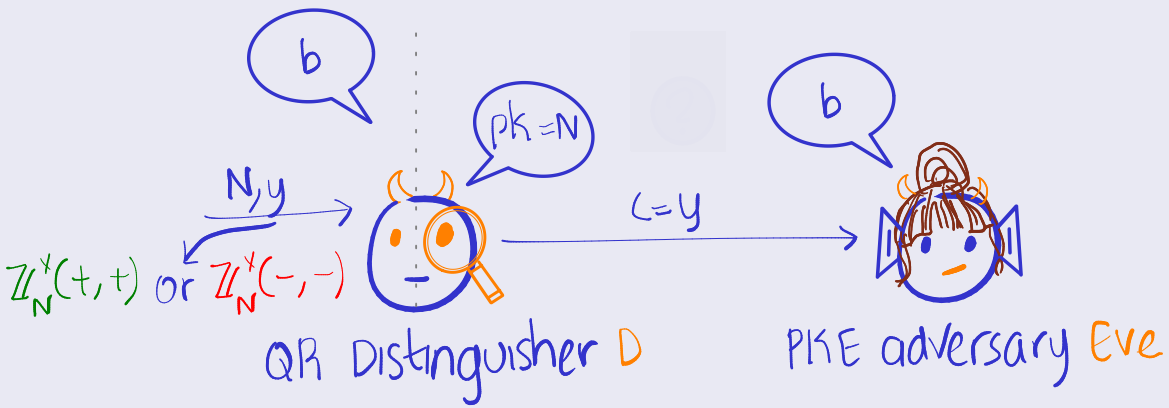

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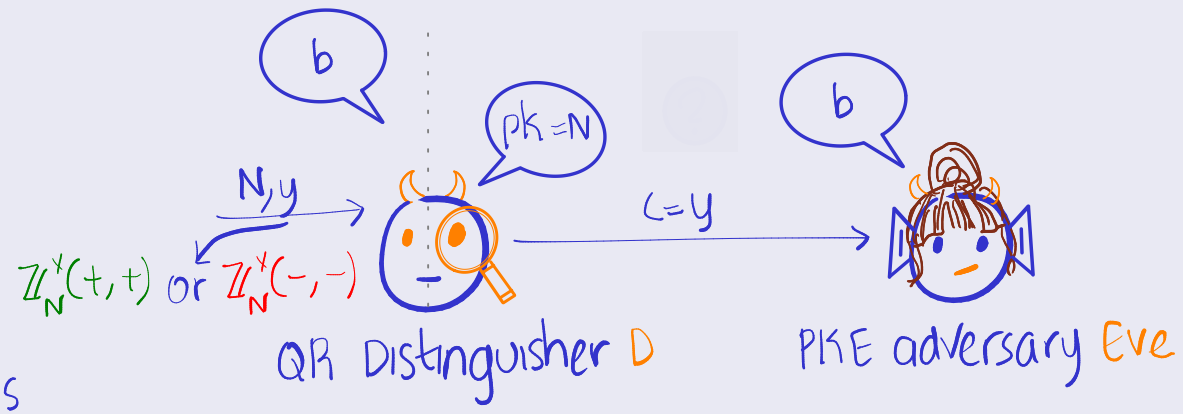

Analysis

$$\left| \Pr_{\substack{N \leftarrow S(1^n) \\ y \leftarrow \mathbb{Z}_N^{\times}(+,+)}} [D(N,y)=0] - \Pr_{\substack{N \leftarrow S(1^n) \\ y \leftarrow \mathbb{Z}_N^{\times}(-,-)}} [D(N,y)=0] \right|$$

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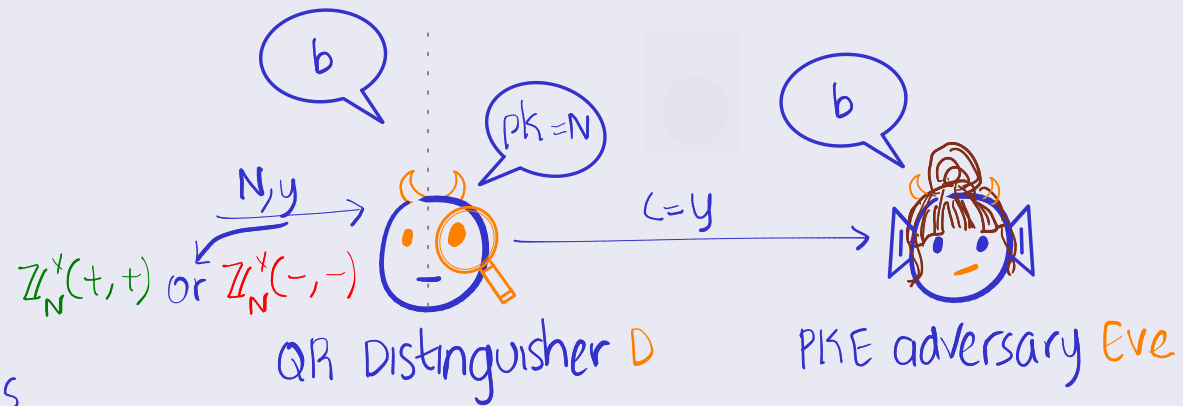

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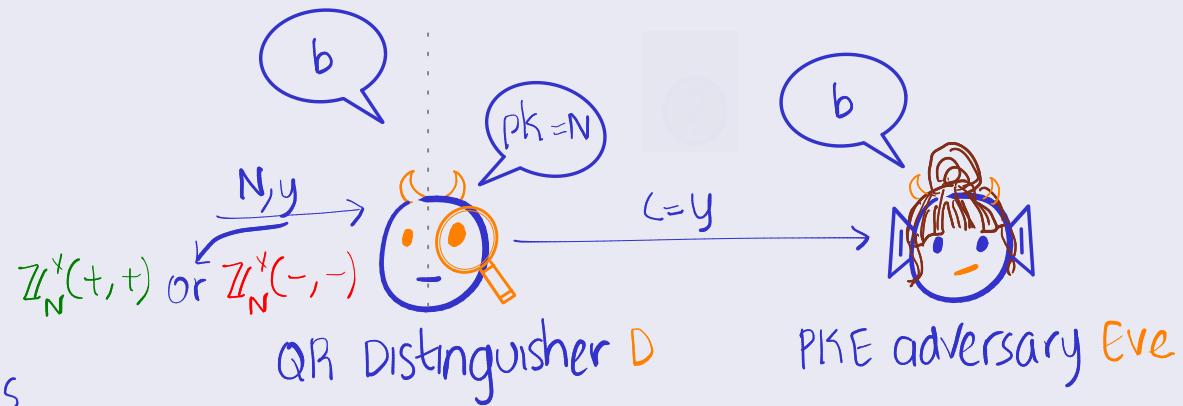

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distributed identically to Enc(pk, 1)

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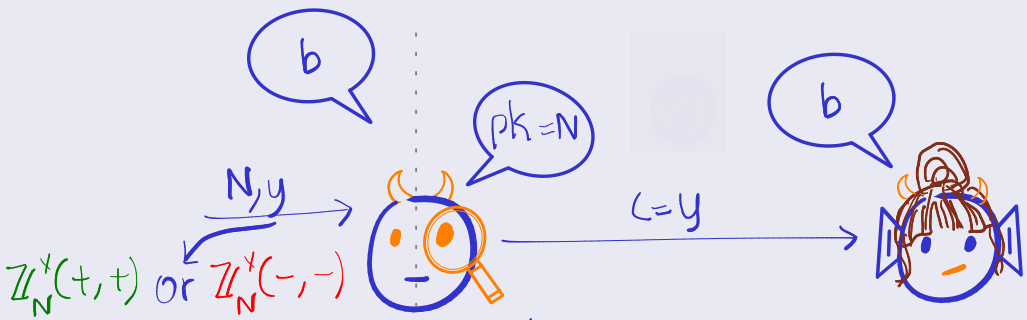

$\Rightarrow$ non-negligible

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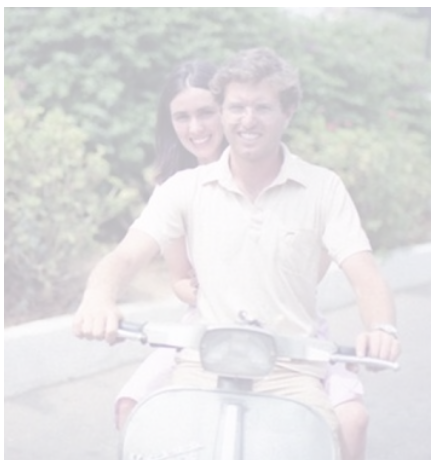

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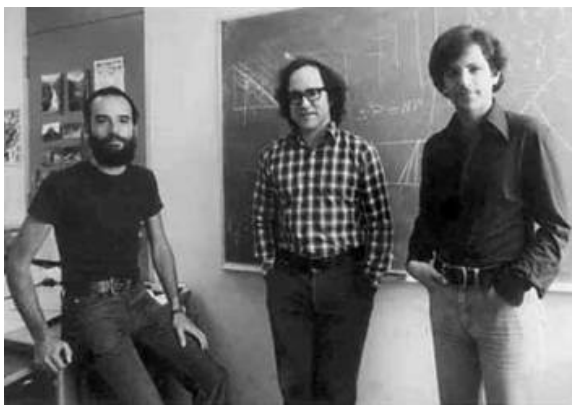
RSA PKE



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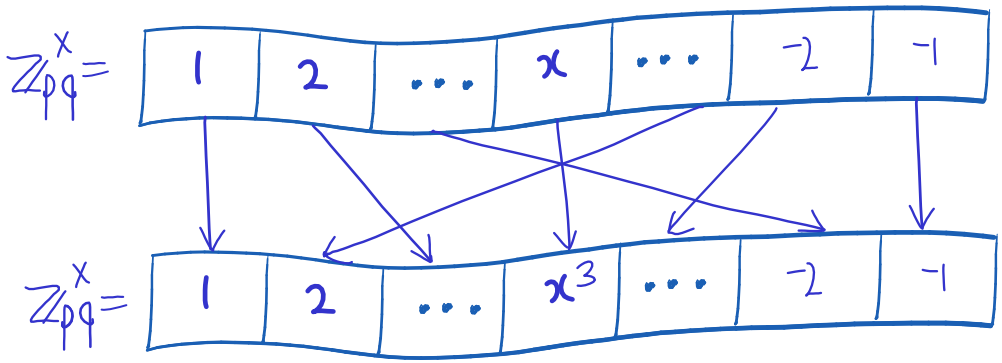
Algebraic setting: multiplication modulo semiprime (RSA group)

Powering Map $x \mapsto x^e \bmod pq$



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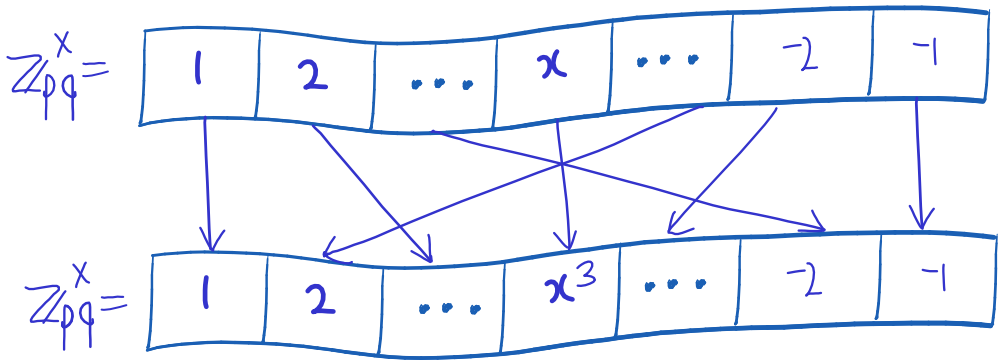

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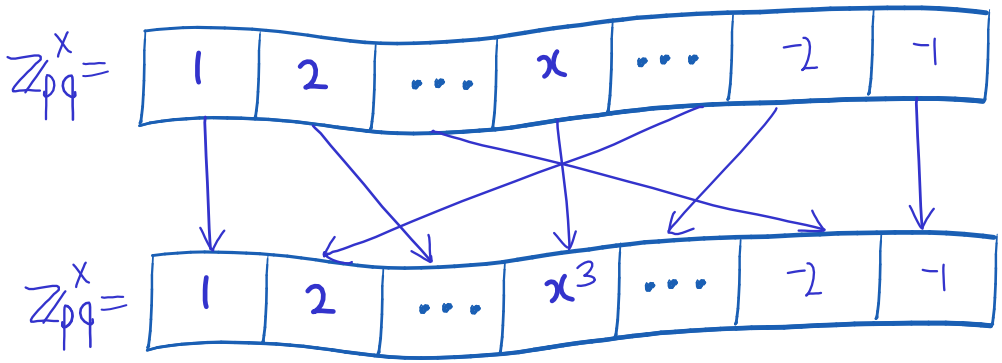

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(Textbook) RSA PKE


Key idea: apply the power map to encrypt

Pseudocode 4 (RSA PKE for $\mathcal{M}_n := \mathbb{Z}_N^\times$)

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  4. Output $(pk := (N, e), sk := (N, d))$
- Encryption $\mathsf{Enc}(pk, m)$: Output $c := m^e \bmod N$
- Decryption $\mathsf{Dec}(sk, c)$: Output $m := c^d \bmod N$

(Textbook) RSA PKE


Key idea: apply the power map to encrypt

Pseudocode 4 (RSA PKE for $\mathcal{M}_n := \mathbb{Z}_N^\times$)

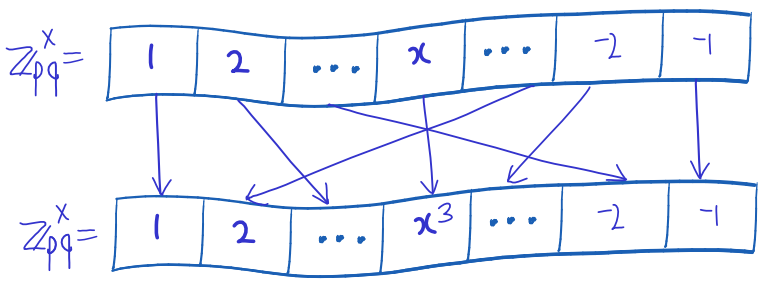
- Key generation $\mathsf{Gen}(1^n)$:
  1. Sample semiprime with factors: $(N, (p, q)) \leftarrow \mathsf{S}(1^n)$
  2. Sample $e \leftarrow [1, \phi(N)]$ such that $gcd(e, \phi(N)) = 1$
  3. Compute d such that $ed = 1 \bmod \phi(N)$
  4. Output $(pk := (N, e), sk := (N, d))$
- Encryption $\mathsf{Enc}(pk, m)$: Output $c := m^e \bmod N$
- Decryption $\mathsf{Dec}(sk, c)$: Output $m := c^d \bmod N$
- Correctness of decryption : $\forall m \in \mathbb{Z}_N^\times : (m^e)^d = m^{ed} = m \bmod N$

Exercise 3

- Show that RSA PKE is not IND-CPA secure
- Show that RSA PKE is OW-CPA secure (see Assignment 4)

Recap/Next Lecture

- Group-based functions (easy) vs. their inverse (hard):
  - $\mathbb{Z}_p^\times$: exponentiation ($x \mapsto g^x$) vs discrete-log
  - $\mathbb{Z}_N^\times$: squaring ($x \mapsto x^2 \bmod pq$) vs square root
  - $\mathbb{Z}_N^\times$: e-th power ($x \mapsto x^e \bmod pq$) vs e-th root
- Built PKE/KEx based on these hard problems
- Takeaway: structure, structure, structure



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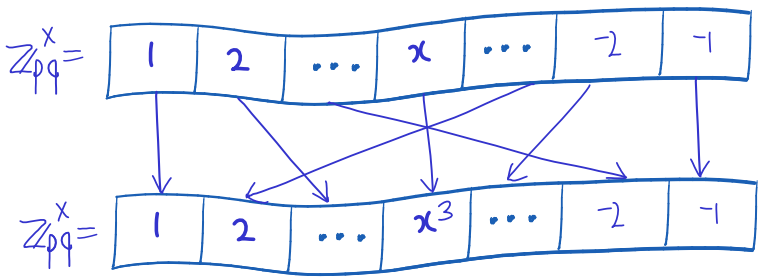

- Next Lecture(s): how to deal with active adversary?
  - Digital signatures: public-key version of MAC
  - How to construct digital signature

References

1. [KL14, Chapter 9.2] for more on the number theory used in this lecture
2. Goldwasser-Micali PKE was proposed in [GM82]. That paper is considered to be the first paper on "provable security"
3. A description of the RSA PKE can be found in [KL14, Chapter 12.5]

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