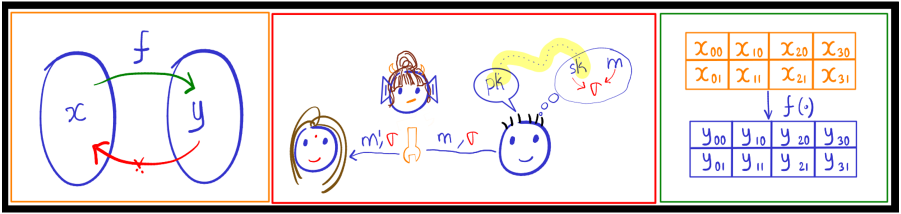


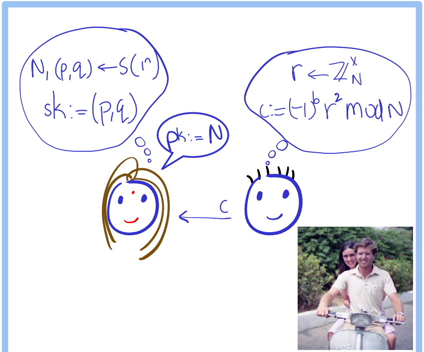
CS409m: Introduction to Cryptography

Lecture 15 (08/Oct/25)

Instructor: Chethan Kamath

- Tasks: Public-key encryption (PKE)
- Threat model: IND-CPA

Goldwasser-Micali PKE



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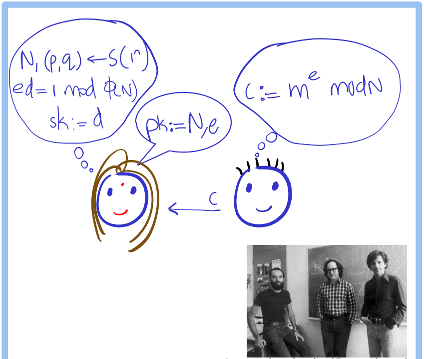
RSA PKE

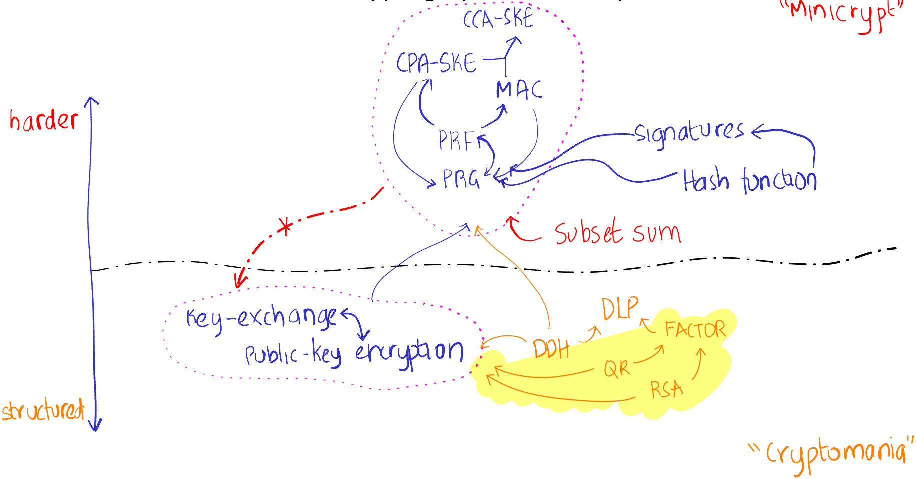


©cs.miami.edu (Rosenberg)

Algebraic setting: multiplication modulo *semiprime* (RSA group)

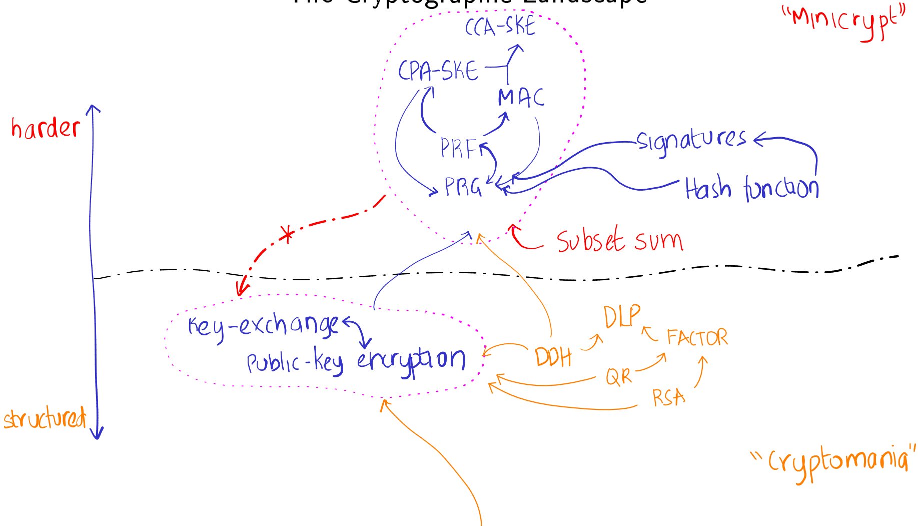
Recall from Last Lecture...

The Cryptographic Landscape



Hardness assumptions: integer factoring, QR and RSA

The Cryptographic Landscape



Elliptic curves, lattices, isogenies, class groups...

Plan for Today's Lecture...

- Task: integrity and authentication in the *public-key* setting
- Threat model: EU-CMA

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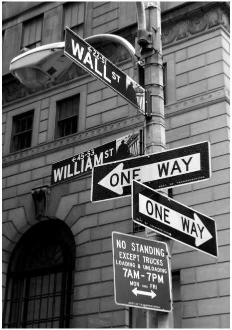
NEW Digital Signature

One-Way Function NEW



Proof technique: plug and pray

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©Lubos Houska

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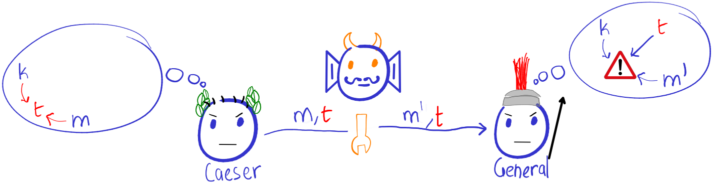
Digital (Analogues of Physical) Signatures . . .

- Recall: Message-Authentication Code (MAC)
- Used to detect tampering by active adversary



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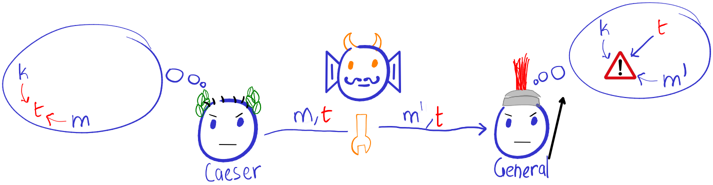

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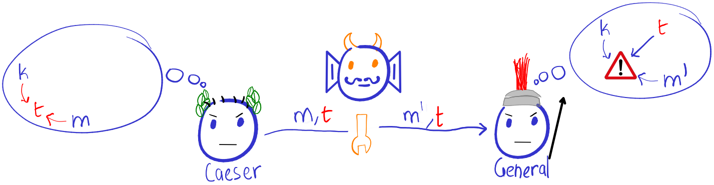

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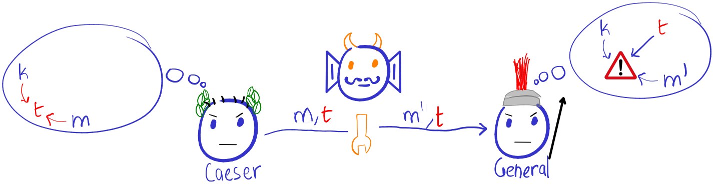

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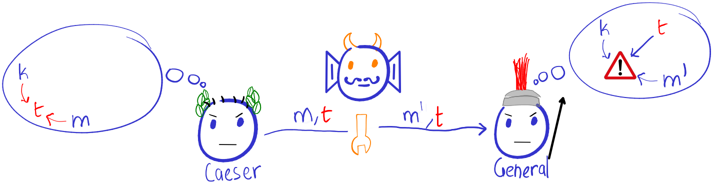

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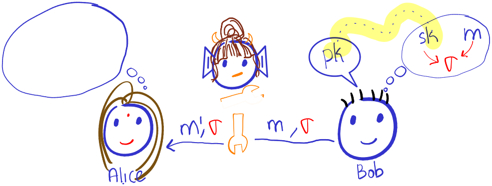

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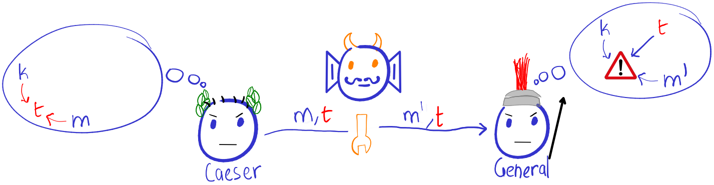

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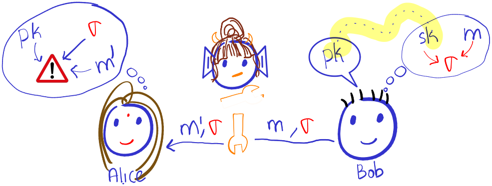

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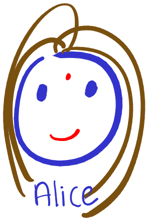

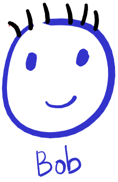

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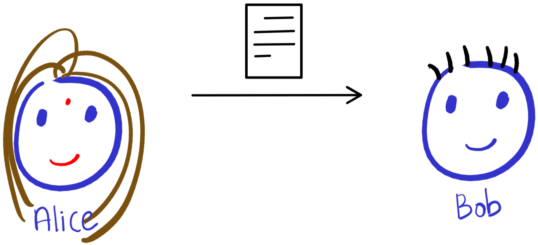

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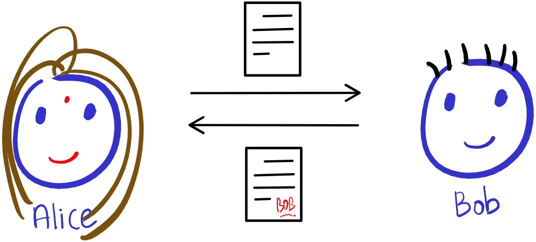

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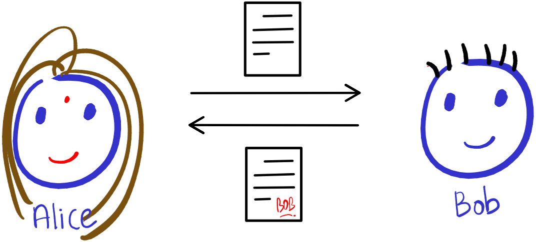

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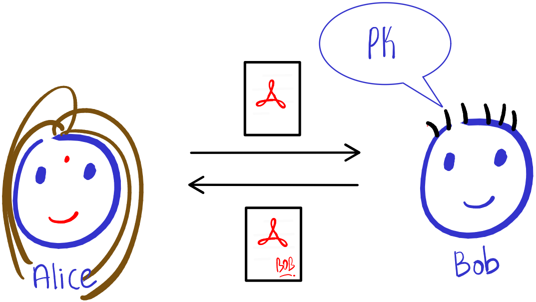

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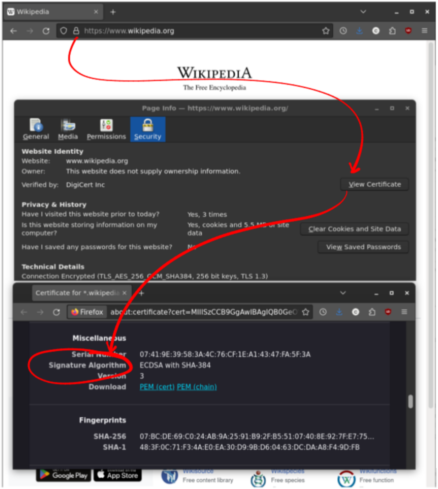

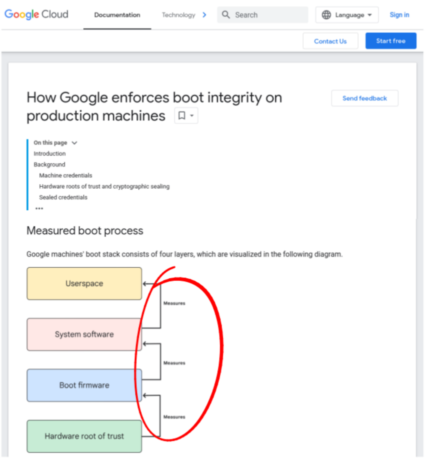

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Digital Signatures: Syntax

- *Public-key* analogue of message authentication codes (MAC)

Definition 1 (Digital signature (DS))

A DS Σ is a triple of efficient algorithms $(\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver})$ with the following syntax:

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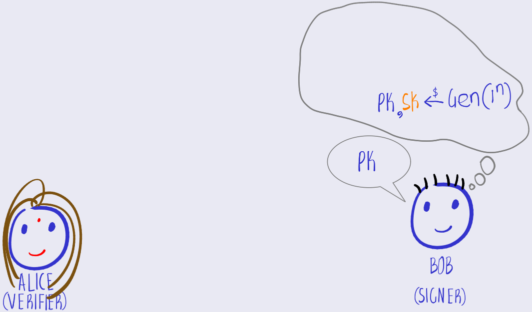

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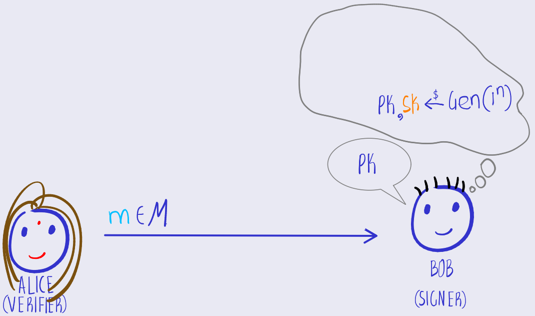

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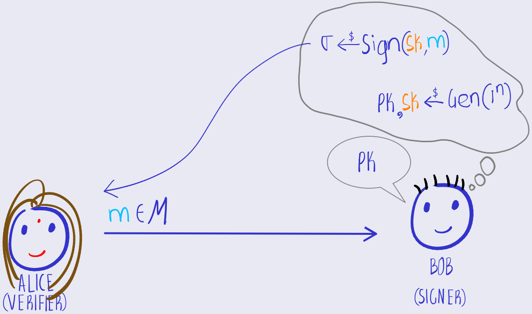

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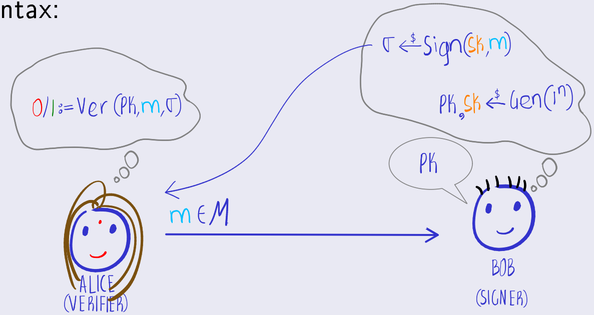

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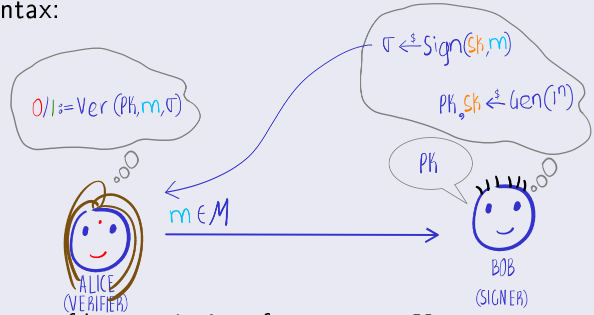

- Correctness of honest signing: for every $n \in \mathbb{N}$, message $m \in \mathcal{M}_n$,

$$\Pr_{(pk,sk)\leftarrow \mathsf{Gen}(1^n),\sigma\leftarrow \mathsf{Sign}(sk,m)}[\mathsf{Ver}(pk,\sigma,m)=1]=1$$

How to Define Security?...

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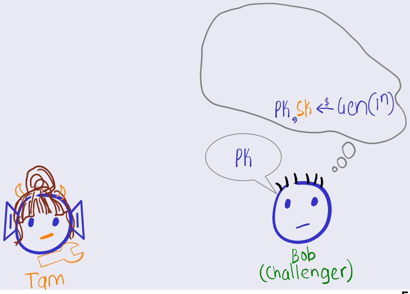

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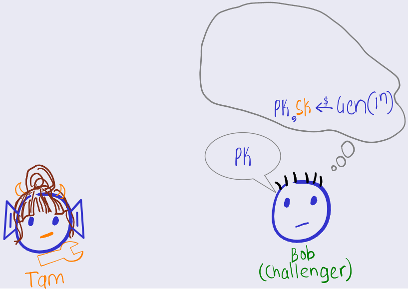

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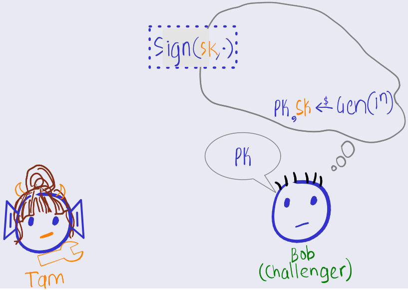

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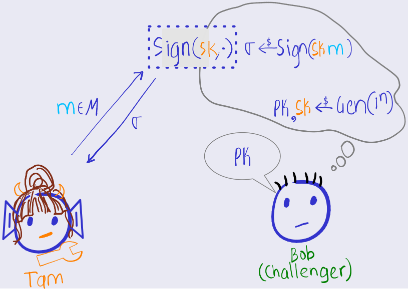

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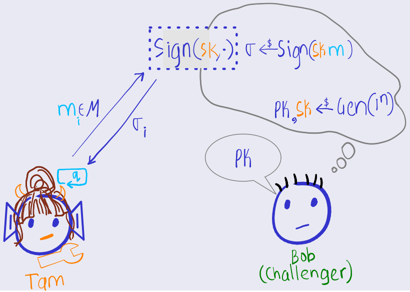

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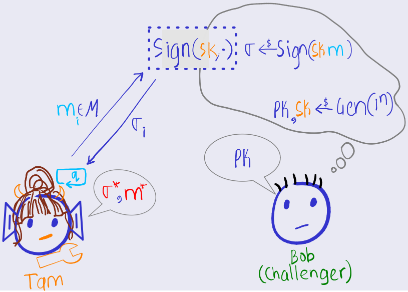
- Tam makes q queries to Sign(sk, ·) oracle
- In the end Tam outputs (σ^*, m^*) and breaks Σ if:
  - $\mathsf{Ver}(pk, m^*, \sigma^*) = 1$
  - $\forall i \in [q] : m^* \neq m_i$



❓ Σ' EU-CMA Secure or Not?

$\Sigma = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Ver}) \to \Sigma' = (\mathsf{Gen}', \mathsf{Sign}', \mathsf{Ver}')$

1. Truncate-then-sign: define Σ' as
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Exercise 1

Prove by reduction that the Σ's in 1 and 3 are EU-CMA-secure.

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- Threat model: EU-CMA

NEW Digital Signature


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One-Way Function NEW

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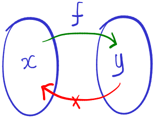
Proof technique: plug and pray

Let's Define One-Way Functions...

- Intuitively: "easy to compute" function f that is "hard to invert"



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"Solution"

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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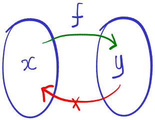

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
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"Puzzle"

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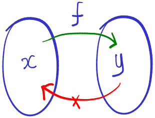

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				8			7	9

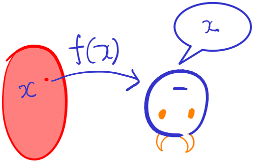
"Puzzle"

- What does "hard to invert" entail? Attempt 1 :

$\forall$ PPT inverter Inv, $\forall x$,

$$\Pr[\mathrm{Inv}(f(x)) = x]$$

is negligible.



- Problem:

Let's Define One-Way Functions...

Intuitively: "easy to compute" function f that is "hard to invert"

"Solution"

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6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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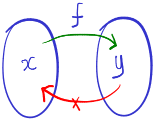

5	3			7				
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	9	8					6	
8				6				3
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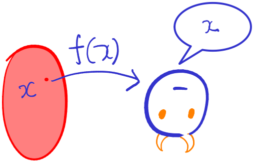

- Problem: Too much to ask (everywhere hardness)

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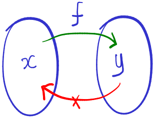

5	3			7				
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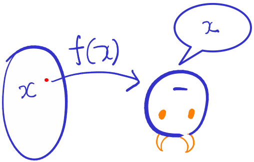

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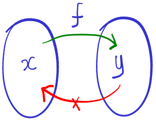

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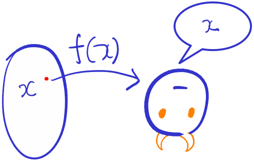

- Problem: This is not sufficient (worst-case hardness)

Let's Define One-Way Functions...

- Intuitively: "easy to compute" function f that is "hard to invert"

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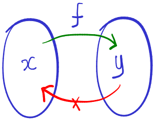
"Solution"



5	3			7				
6			1	9	5			
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8				6				3
4			8		3			1
7				2				6
	6					2	8	
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"Puzzle"

- What does "hard to invert" entail? Attempt 3 :

∀ PPT inverter Inv

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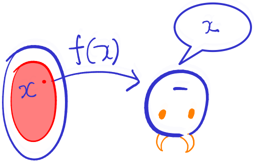

- Problem:

Let's Define One-Way Functions...

Intuitively: "easy to compute" function f that is "hard to invert"

"Solution"

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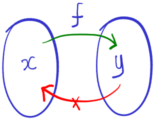

5	3			7				
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8				6				3
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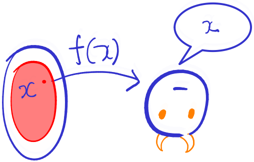

- Problem: What about $f(x) := 0^{|x|}$?

Intuitively: "easy to compute" function f that is "hard to invert"

"Solution"

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6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
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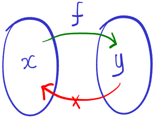

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"Puzzle"

- What does "hard to invert" entail? Attempt 4 :

$\forall$ PPT inverter Inv

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathrm{Inv}(f(x)) \in f^{-1}(f(x))\right]$$

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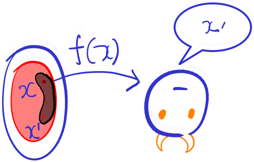

- Problem:

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"Solution"

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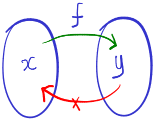
©Cburnett/Wikipedia



5	3			7				
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8				6				3
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7				2				6
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"Puzzle"

©Tim Stellmach/Wikipedia

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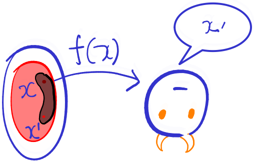


- Problem: ?

Let's Define One-Way Functions...

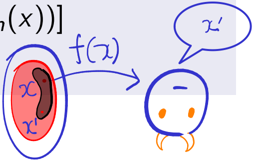
Intuitively: "easy to compute" function that is "hard to invert"

Definition 3 (One-way function (OWF))

A function family $f := \left\{f_n : \{0,1\}^n \to \{0,1\}^{m(n)}\right\}_{n\in\mathbb{N}}$ is one-way if

- there exists an efficient algorithm F such that $\forall x : \mathsf{F}(x) = f(x)$
- for all PPT *inverters* Inv, the following is negligible:

$$p(n) := \Pr_{x \leftarrow \{0,1\}^n}[\mathsf{Inv}(f_n(x)) \in f_n^{-1}(f_n(x))]$$



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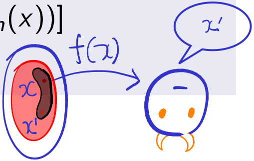

- Length-preserving OWF: $m(n) = n$
- One-way permutation: f is length-preserving and *bijective*

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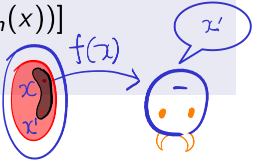

- Length-preserving OWF: $m(n) = n$
- One-way permutation: f is length-preserving and *bijective*
- Convenient to consider "collection" of OWF:

$$\{f_I : \mathcal{D}_I \to \mathcal{R}_I\}_{I\subseteq\{0,1\}^*}$$

OWF or Not?

- Some generic constructions:
  1. $f_1(x) := f(x) \| 0^{|x|}$, where f is a OWF
  2. $f_2(x_1 \| x_2) := x_1 \| f(x_2)$, where f is a OWF and $|x_1| - |x_2| \leq 1$
  3. $f_3(x_1 \| x_2) := x_1 \| f(x_1 \| x_2)$, where f is a OWF and $|x_1| - |x_2| \leq 1$
  4. $f_4(x) := \mathsf{G}(x)$, where G is a PRG

OWF or Not?

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- A concrete construction:
  4. $f_5(x_1 \| x_2) := x_1 \cdot x_2$, where x_1 and x_2 are parsed as integers

OWF or Not?

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OWF or Not?

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     - "Weakly" one-way since primes are dense enough

Exercise 2

1. Show using security reduction that f_1, f_2 and f_4 are OWFs
2. Come up fs such that f_3 i) remains one-way and ii) becomes invertible

We've Already Seen Some OWF Collections!

large constant in $\mathbb{Z}_p^*$

1. Multiplication modulo prime p: $f_{p,c}(x) := cx \bmod p$

We've Already Seen Some OWF Collections!

large constant in $\mathbb{Z}_p^*$

1. Multiplication modulo prime p: $f_{p,c}(x) := cx \bmod p$
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   $n \times m$ matrix over $\mathbb{Z}_p^*$

We've Already Seen Some OWF Collections!

large   constant in $\mathbb{Z}_p^*$

1. Multiplication modulo prime p: $f_{p,c}(x) := cx \bmod p$
2. Matrix multiplication modulo prime p: $f_{\overline{A}}(\overline{x}) := \overline{x}^T \overline{A} \bmod p$ $n \times m$ matrix over $\mathbb{Z}_p^*$
   - Inversion easy by Gaussian elimination
3. Squaring modulo prime p: $f_p(x) := x^2 \bmod p$
4. Squaring modulo semiprime $N = pq$: $f_N(x) := x^2 \bmod N$

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large   constant in $\mathbb{Z}_p^*$

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2. Matrix multiplication modulo prime p: $f_{\overline{A}}(\overline{x}) := \overline{x}^T\overline{A} \bmod p$
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   - Inversion as hard as factoring N

   generator
5. Exponentiation modulo prime p: $f_{p,g}(x) := g^x \bmod p$

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large — constant in $\mathbb{Z}_p^*$

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2. Matrix multiplication modulo prime p: $f_{\overline{A}}(\overline{x}) := \overline{x}^T \overline{A} \bmod p$ — $n \times m$ matrix over $\mathbb{Z}_p^*$
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4. Squaring modulo semiprime $N = pq$: $f_N(x) := x^2 \bmod N$
   - Inversion as hard as factoring N
5. Exponentiation modulo prime p: $f_{p,g}(x) := g^x \bmod p$ — generator
   - Inversion is the Discrete Logarithm Problem: believed hard
6. Power map modulo semiprime $N = pq$: $f_{N,e}(x) := x^e \bmod N$

We've Already Seen Some OWF Collections!

large constant in $\mathbb{Z}_p^*$

1. Multiplication modulo prime p: $f_{p,c}(x) := cx \bmod p$
2. Matrix multiplication modulo prime p: $f_{\bar{A}}(\bar{x}) := \bar{x}^T \bar{A} \bmod p$
   - Inversion easy by Gaussian elimination $n \times m$ matrix over $\mathbb{Z}_p^*$
3. Squaring modulo prime p: $f_p(x) := x^2 \bmod p$
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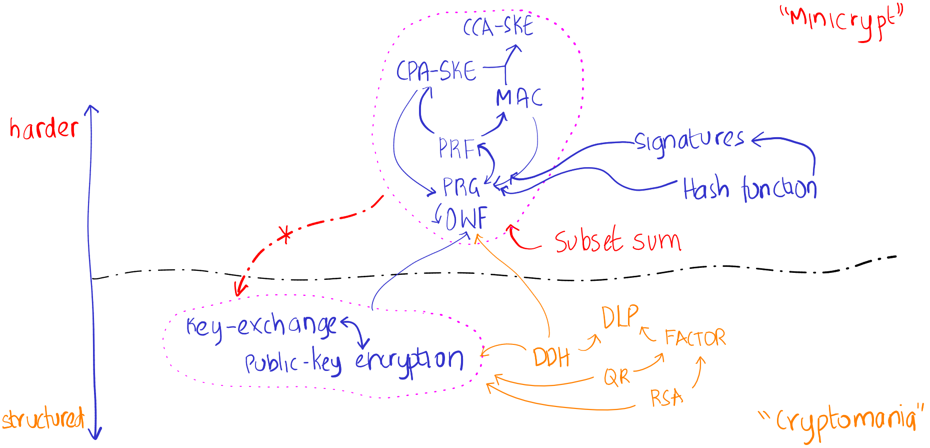
   - Inversion is the Discrete Logarithm Problem: believed hard
6. Power map modulo semiprime $N = pq$: $f_{N,e}(x) := x^e \bmod N$
   - Inversion is the RSA problem: believed hard

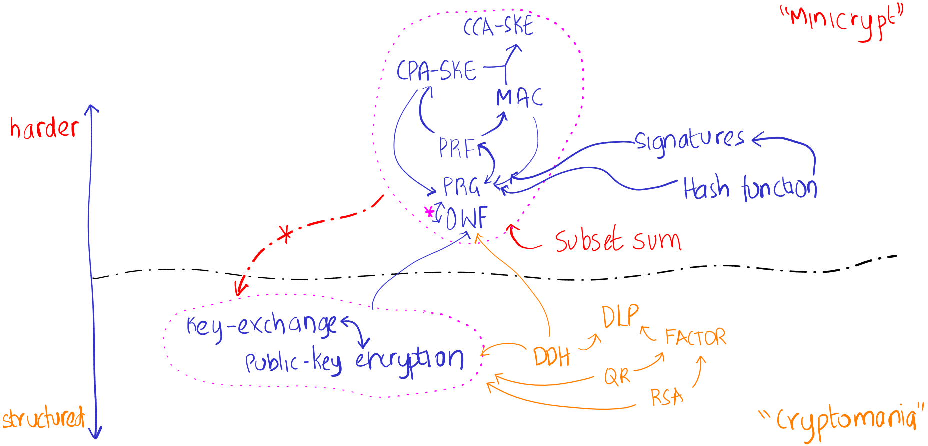
Exercise 3

Show that taking square root modulo N is equivalent to factoring N.
(Hint: use the identity $x^2 - y^2 = (x + y)(x - y) \bmod N$)

The Cryptographic Landscape



The Cryptographic Landscape



* **Theorem 1 ([HILL99, BM82])**

If one-way functions exist then so do pseudo-random generators

Plan for Today's Lecture...

- Task: integrity and authentication in the *public-key* setting
- Threat model: EU-CMA

NEW Digital Signature

©Wikipedia

One-Way Function NEW

©Lubos Houska

Proof technique: plug and pray

One-Time DS ($q = 1$): Lamport's Signature

$\{f_n : \{0,1\}^n \to \{0,1\}^n\}_n$

Construction 1 (OWF $f \to$ one-time DS Σ for $\mathcal{M} := \{0,1\}^\ell$)

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$\{f_n : \{0,1\}^n \to \{0,1\}^n\}_n$

Construction 1 (OWF f → one-time DS Σ for $\mathcal{M} := \{0,1\}^{\ell}$) $^{=4}$

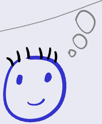
$x_{ib} \leftarrow \{0,1\}^n$

x_{00}	x_{10}	x_{20}	x_{30}
x_{01}	x_{11}	x_{21}	x_{31}

$= sk$

$Gen(1^n)$

ALICE
(VERIFIER)



BOB
(SIGNER)

One-Time DS ($q = 1$): Lamport's Signature

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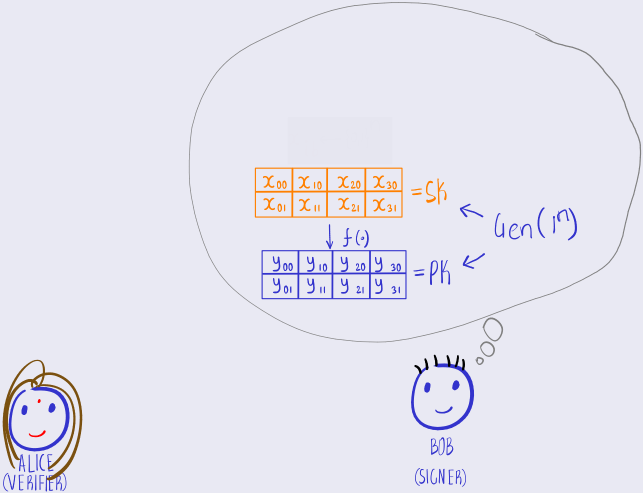
Construction 1 (OWF f $\to$ one-time DS Σ for $\mathcal{M} := \{0,1\}^{\ell}$) $=4$



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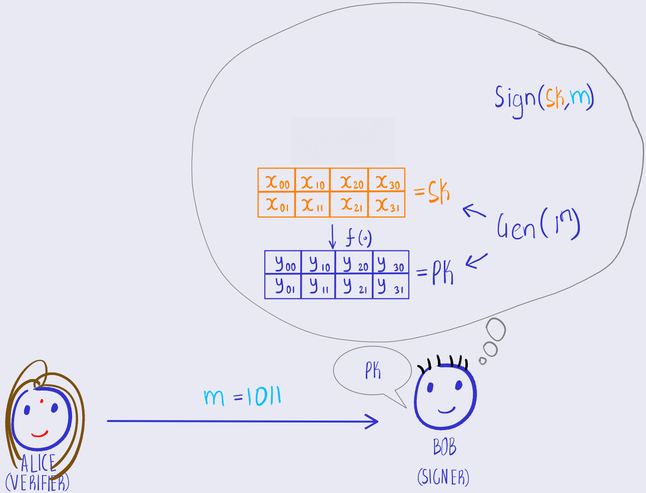

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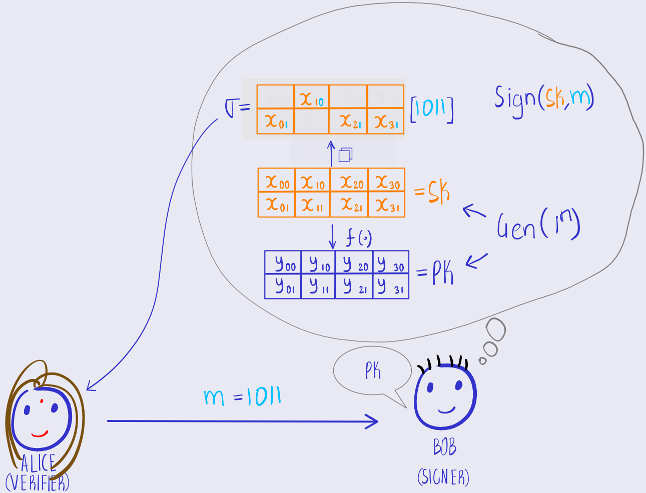

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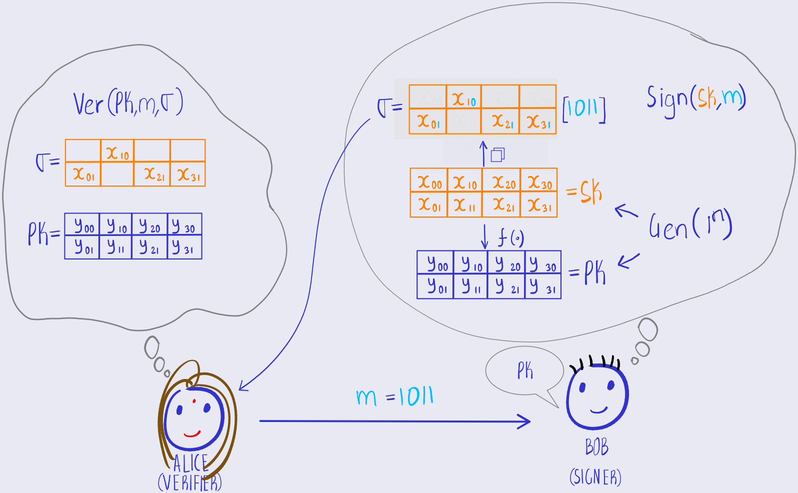

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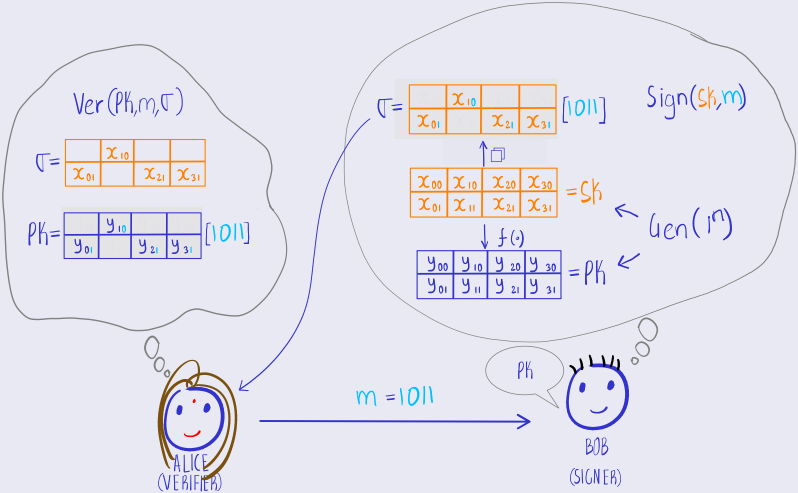

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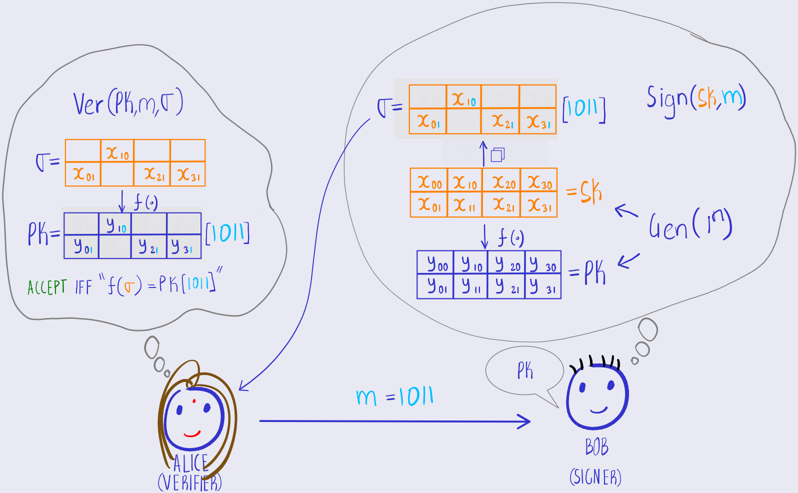

Lamport's Signature is One-Time Secure...

Theorem 2

If f is a OWF then Lamport's scheme is a one-time DS.

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Proof sketch: proof by reduction. Idea: "plug and pray".

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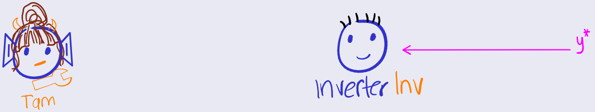

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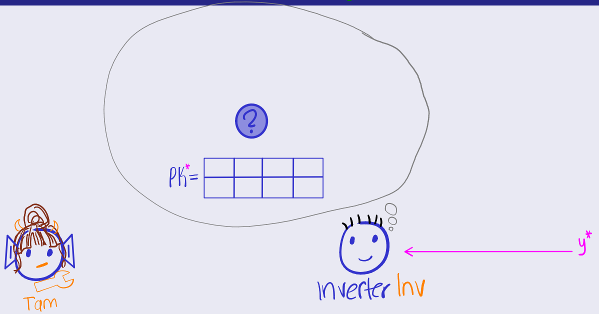

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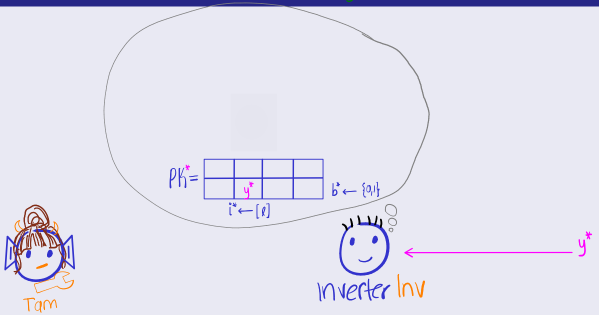

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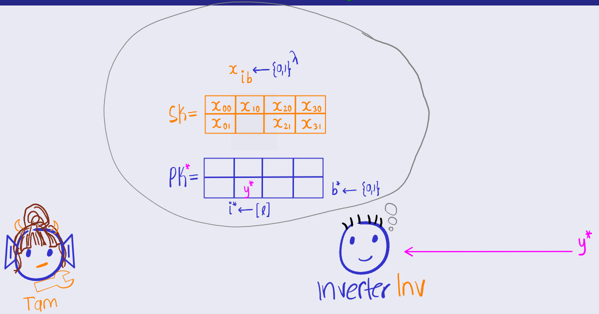

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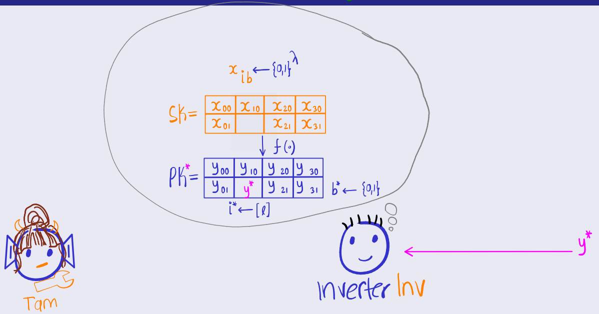

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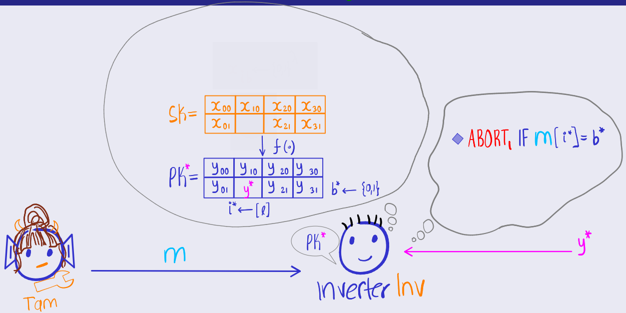

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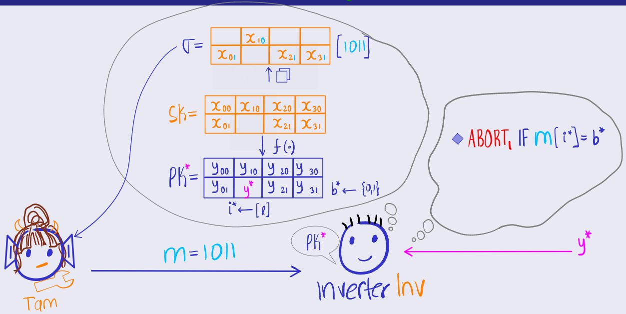

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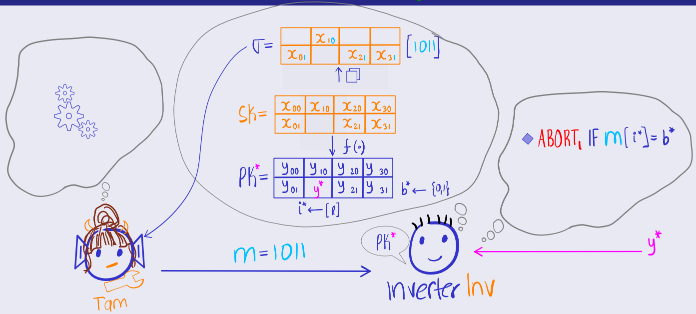

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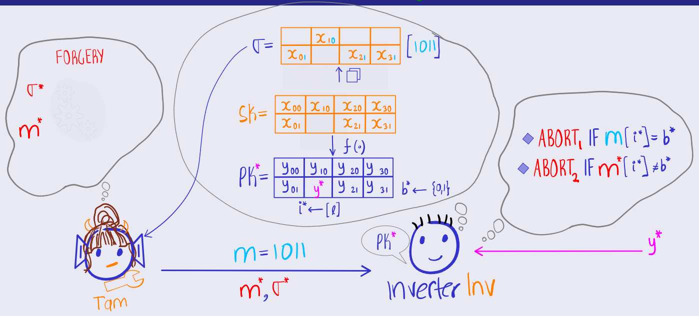

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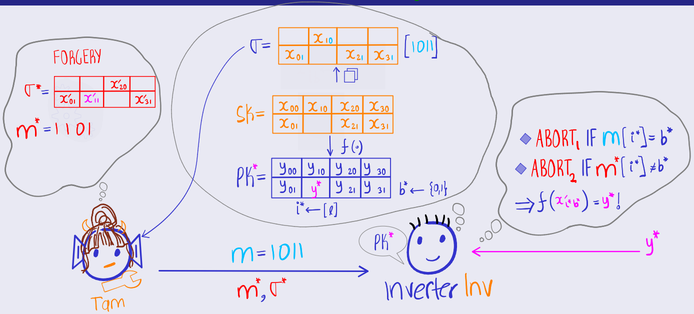

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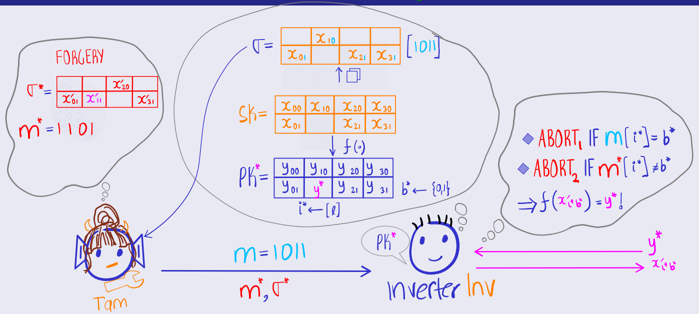

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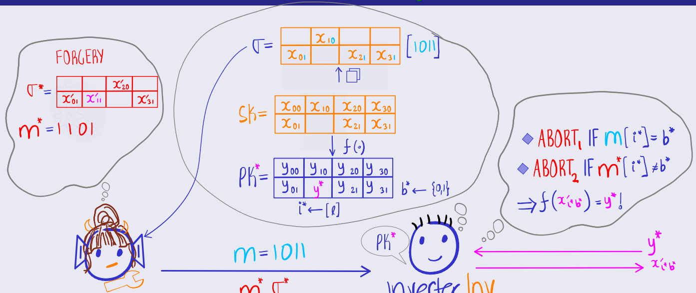

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$$\Pr_{x}[\text{Inv inverts } f(x)] = \Pr_{x,i^*,b^*,sk,m,m^*}\left[\overline{\text{ABORT}_2}, \text{Tam breaks}, \overline{\text{ABORT}_1}\right]$$

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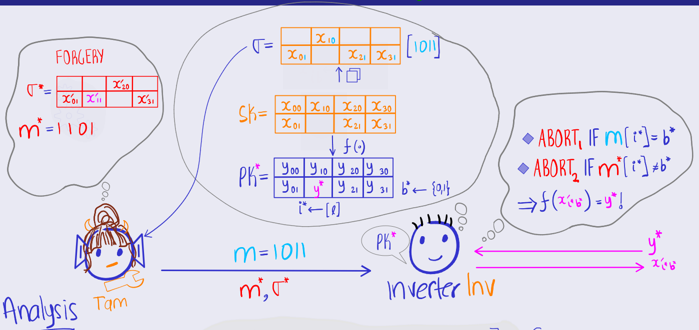

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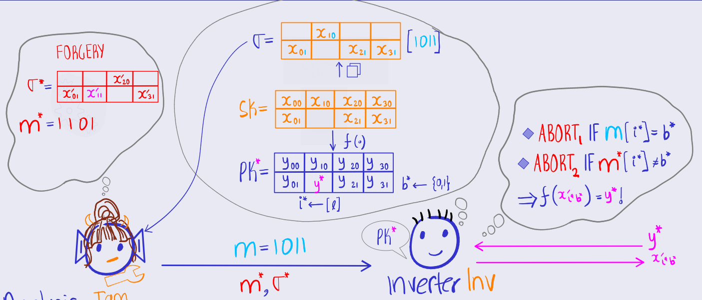

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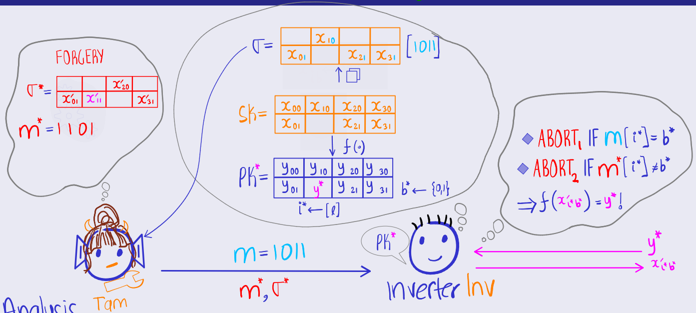

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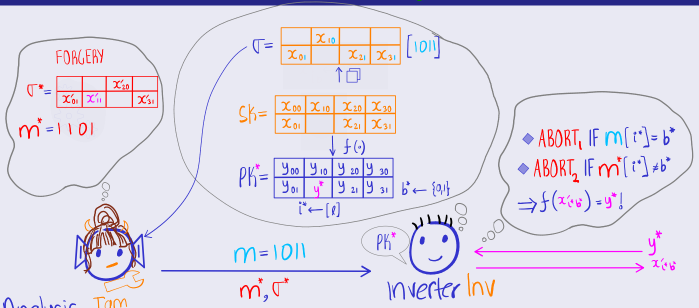
Theorem 2

If f is a quantum-secure OWF then Lamport's scheme is a quantum-secure one-time DS.

Proof sketch: proof by reduction. Idea: "plug and pray".



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Theorem 3

If f is a OWF then Lamport's scheme is a one-time DS for fixed-length messages.

Exercise 5 (Domain Extension)

Given a compressing function $H : \{0,1\}^{2\ell} \to \{0,1\}^{\ell}$, construct a one-time DS for *arbitrary-length* messages. What are the properties you need from H to ensure that the one-time DS is secure?

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Theorem 4 ([Mer90, Gol87])

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2. Step II: Many-time *stateful* DS $\Rightarrow$ Many-time DS
   - Use PRF to *derandomise* Step I

$\square$

Recap/Next Lecture

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- Theoretical constructions of DS
  - Lamport's one-time DS
  - Generic transformation from one-time to many-time DS
  - Takeaway: "Plug and pray"

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- Next lecture: How to sign longer messages?
  - New primitive: collision-resistant hash functions

References

1. Refer to [KL14, Chapters 13.1 and 13.2] for motivation and definition of DS.
2. The construction of one-time DS and the subsequent generic transform (Theorem 4) can be found in [KL14, Chapter 14.4]
3. For a historical take on OWFs, see [DH76].
4. The construction of PRG from OWF is due to [HILL99], building on the construction of PRF from OWP from [BM82].

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