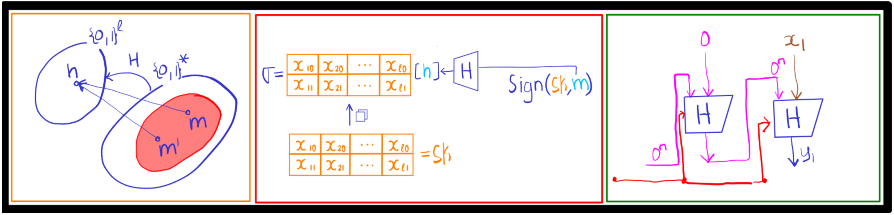


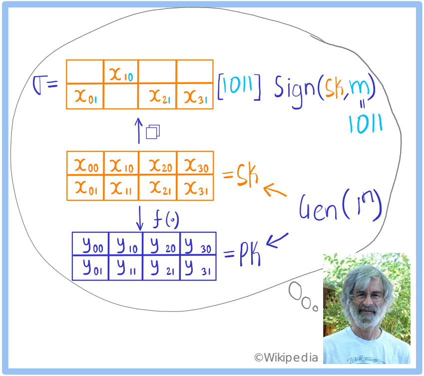
CS409m: Introduction to Cryptography

Lecture 16 (10/Oct/25)

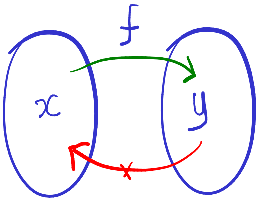
Instructor: Chethan Kamath

- Task: integrity and authentication in the *public-key* setting
- Threat model: EU-CMA

Lamport's Digital Signature



One-Way Function



$$\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{Inv}(f_n(x)) \in f_n^{-1}(f_n(x))]$$

Recall from Last Lecture...

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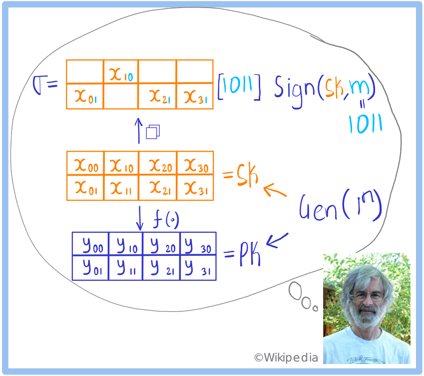

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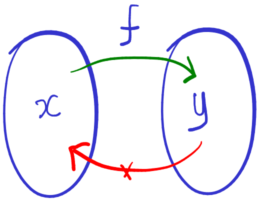

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{Inv}(f_n(x)) \in f_n^{-1}(f_n(x))]$$

Proof technique: plug and pray

Lamport's Signature is One-Time Secure...

Theorem 1

If f is a OWF then Lamport's scheme is a one-time DS

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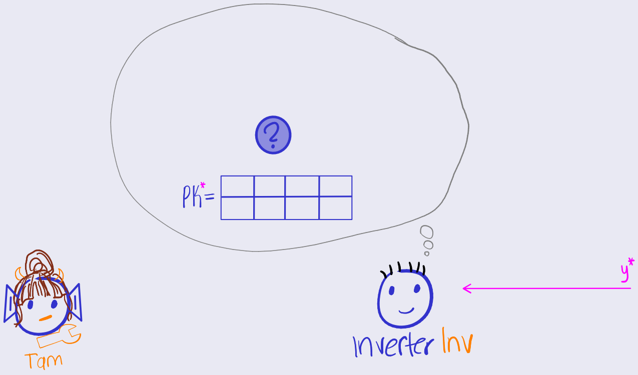

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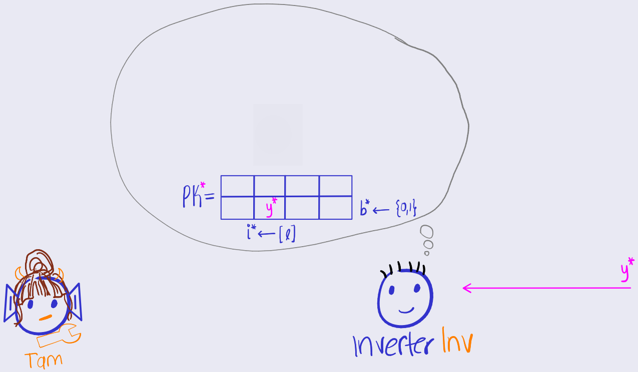

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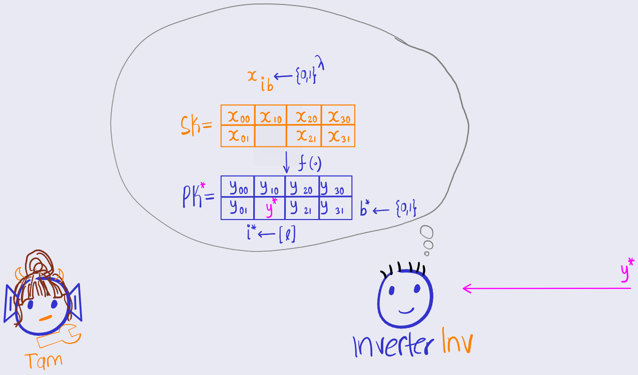

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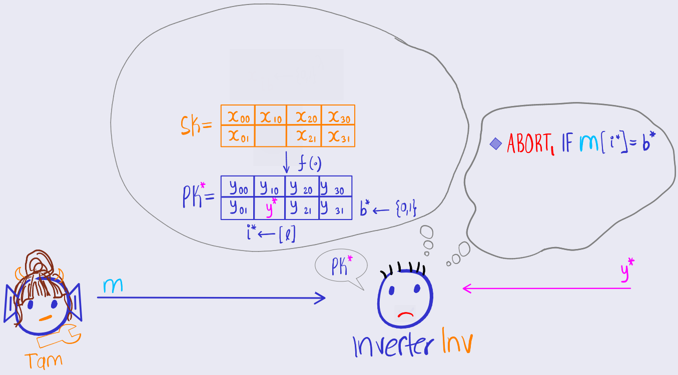

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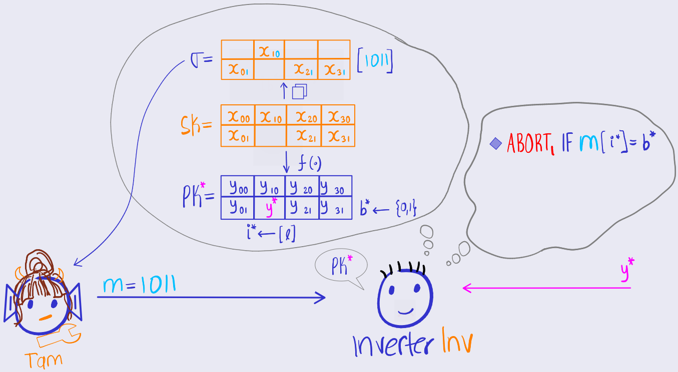

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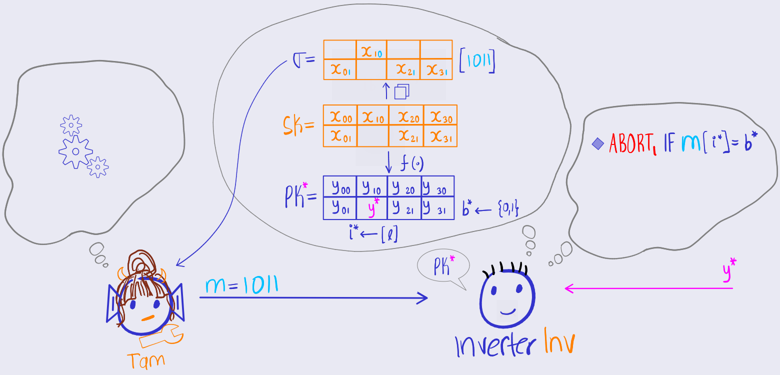

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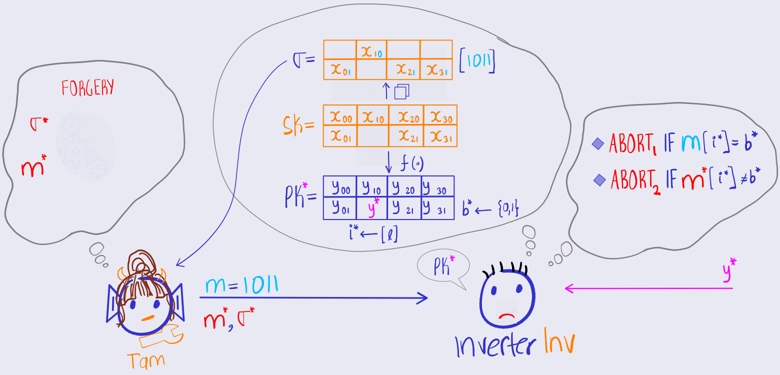

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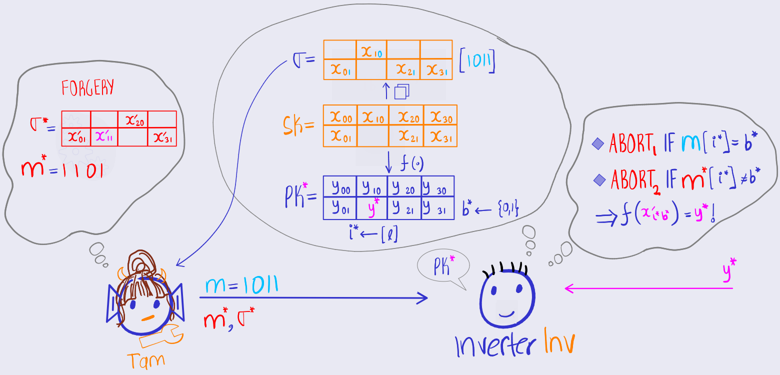

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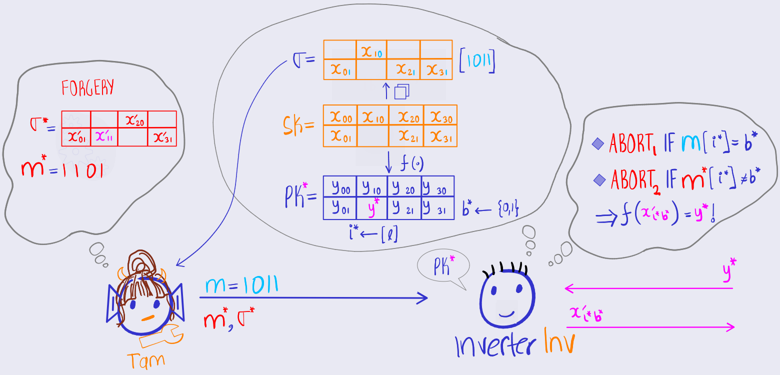

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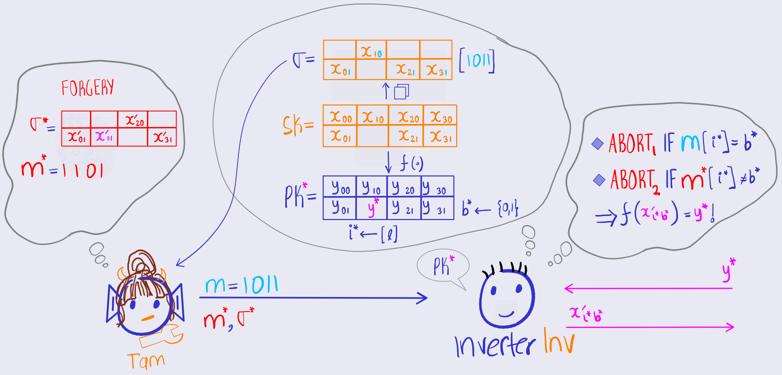

(Analysis on whiteboard)

Lamport's Signature is One-Time Secure...

Theorem 2 ([Mer90a, Gol87])

If one-time DS and PRFs exists then many-time DS exists

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Lamport's Signature is One-Time Secure...

Theorem 2 ([Mer90a, Gol87])

If one-time DS and PRFs exists then many-time DS exists

Theorem 1

If f is a OWF then Lamport's scheme is a one-time DS for fixed-length messages!

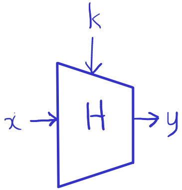
Exercise 1 (Domain Extension)

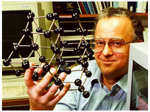
Given a compressing function $H : \{0,1\}^{2\ell} \to \{0,1\}^{\ell}$, construct a one-time DS for *arbitrary-length* messages. What are the properties you need from H to ensure that the one-time DS is secure?

Plan for Today's Lecture...

- Task: sign arbitrarily long messages
- Threat model: EU-CMA

NEW Hash Function





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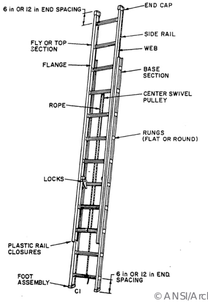
Domain Extension NEW



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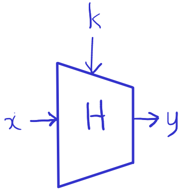

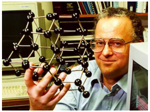

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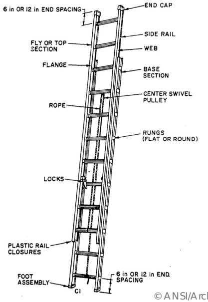

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Old tricks: chain, tree-based constructions

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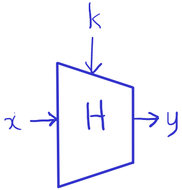
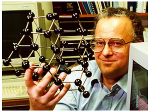
NEW **Hash Function**





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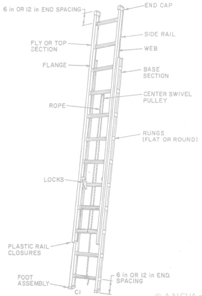

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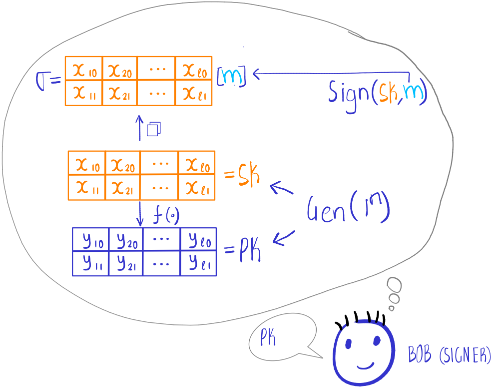
Old tricks: chain, tree-based constructions

Extending Lamport's One-Time DS for Long Messages...



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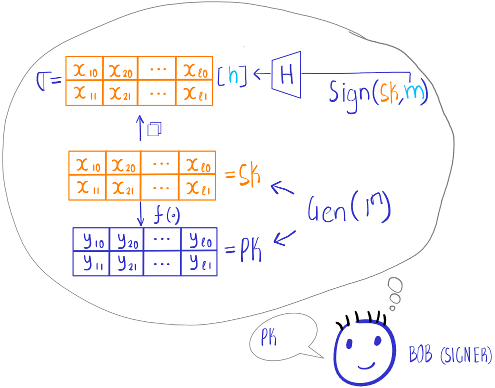
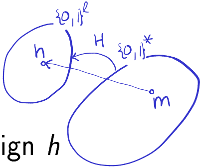
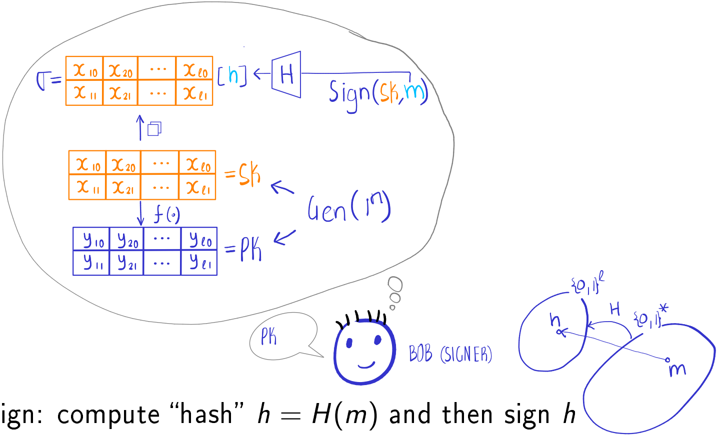
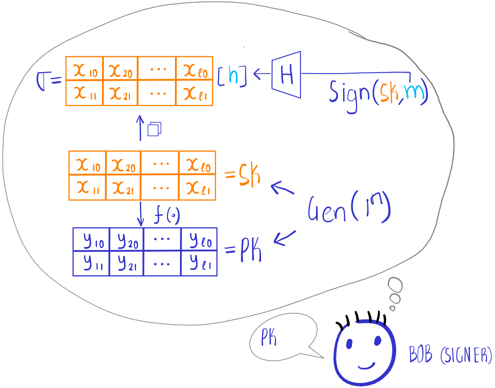


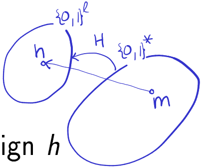


- Hash-then-sign: compute "hash" $h = H(m)$ and then sign h



- Hash-then-sign: compute "hash" $h = H(m)$ and then sign h
- What are the requirements from H? When can Tam forge?





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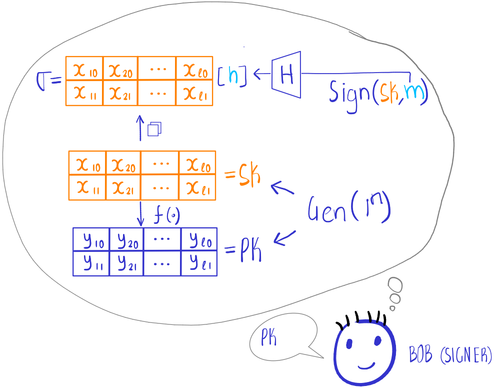

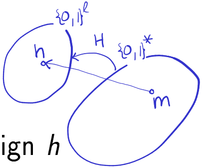

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  - Collisions are guaranteed to exist (pigeonhole principle)

- Is "collision-resistance" sufficient?

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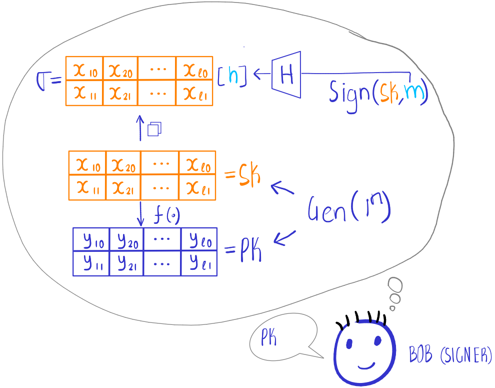

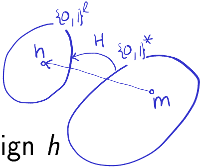

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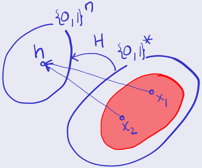
  - Is "collision-resistance" sufficient? Yes, as we'll see.

Definition 1 (CRHF, with key generation algorithm Gen)

A keyed function (family) $\{H : \mathcal{K} \times \{0,1\}^* \to \{0,1\}^n\}$ is a CRHF if for every PPT collision-finder F, the following is negligible.

$$\Pr_{\substack{k \leftarrow \mathrm{Gen}(1^n) \\ (x_1 \overset{\neq}{,} x_2) \leftarrow \mathsf{F}(k)}} [H(k, x_1) = H(k, x_2)]$$

Need not be same length



Collision-Resistant Hash Function (CRHF)

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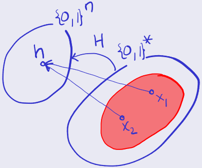

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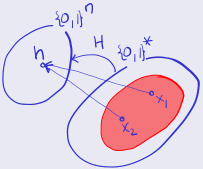

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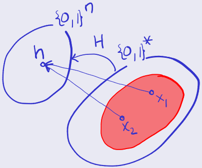

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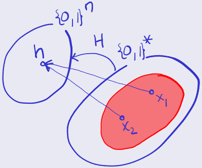

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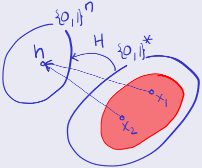

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Exercise 2

Prove formally cases where H is CRHF; describe counter-e.g. otherwise

Let's (Slowly) Find Collisions in H!

What about a *deterministic* $O(2^n)$-time collision-finder?

Let's (Slowly) Find Collisions in H!

What about a *deterministic* $O(2^n)$-time collision-finder?

- Exploit pigeonhole principle

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- Exploit pigeonhole principle
- Compute (e.g.) hash of inputs $0\|0^n, \ldots, 0\|1^n, 1\|0^n$
  - There *must exist* colliding pair of inputs

Can we do better?

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Can we do better? Yes, recall *birthday paradox*:

Theorem 3 (Lecture 2)

Let $q \leq \sqrt{2 \cdot 2^n}$ elements $(y_1, \ldots, y_q)$ be chosen uniformly and independently at random from $\{0,1\}^n$, then

$$\Pr[\exists i \neq j \ s.t. \ y_i = t_j] \geq q(q-1)/42^n$$

How does a *randomised* $O(2^{n/2})$-time collision-finder work?

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How does a *randomised* $O(2^{n/2})$-time collision-finder work?

- Compute hash of $q := O(2^{n/2})$ *random* inputs $x_1, \ldots, x_q \leftarrow \{0,1\}^n$
- By Theorem 3, with *noticeable probability* there exists a colliding pair

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What about a *deterministic* $O(2^n)$-time collision-finder?

Exploit pigeonhole principle

- Compute (e.g.) hash of inputs $0\|0^n, \ldots, 0\|1^n, 1\|0^n$
  - There *must exist* colliding pair of inputs

Can we do better? Yes, recall *birthday paradox*:

Theorem 3 (Lecture 2)

Let $q \leq \sqrt{2 \cdot 2^n}$ elements $(y_1, \ldots, y_q)$ be chosen uniformly and independently at random from $\{0,1\}^n$, then

$$\Pr[\exists i \neq j \ s.t. \ y_i = t_j] \geq q(q-1)/42^n$$

How does a *randomised* $O(2^{n/2})$-time collision-finder work?

- Compute hash of $q := O(2^{n/2})$ *random* inputs $x_1, \ldots, x_q \leftarrow \{0,1\}^n$
- By Theorem 3, with *noticeable probability* there exists a colliding pair

Consequence: key-size/output length must be $2\times$ security level

Extending Lamport's One-Time DS for Long Messages...

Theorem 4

If f is a OWF and H is CRHF then the "hash-then-sign" scheme is a one-time DS for arbitrarily-long messages.

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Collision finder F

Tam

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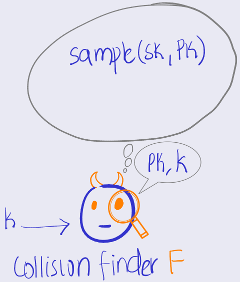

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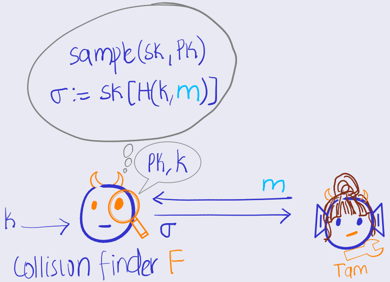

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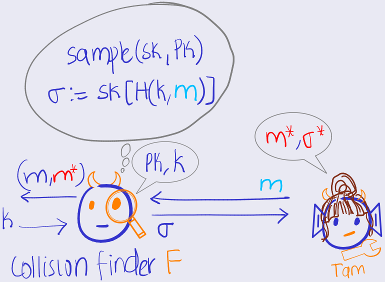

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Inverter Inv


Tam

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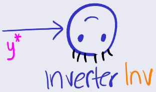

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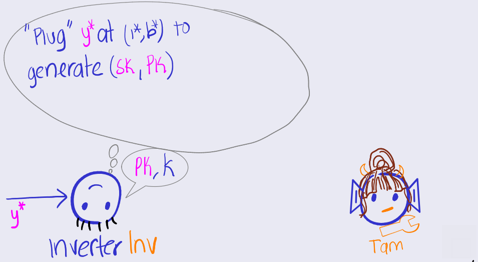

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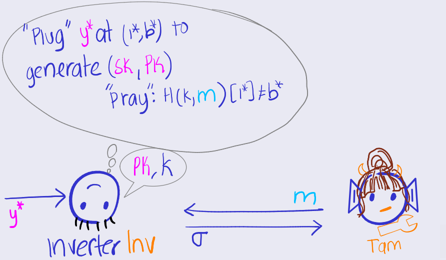

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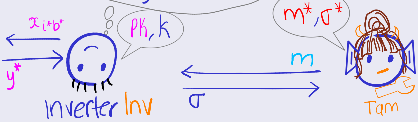
"Plug" y^* at (i^*, b^*) to generate (sk, pk)

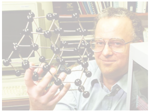
"Pray": $H(k,m)[i^*] \neq b^*$

"Pray": $H(k,m^*)[i^*] = b^*$



- Task: sign arbitrarily long messages
- Threat model: EU-CMA

NEW Hash Function



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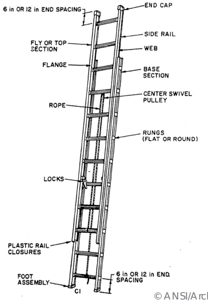
Domain Extension NEW



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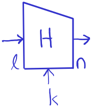
Old tricks: chain, tree-based constructions

Compression Functions and Domain-Extension

- *Compression function*: hash function for fixed input length $\ell(n) > n$
  - Easier to construct in practice: e.g., MD5, SHA2 (unkeyed) compression function of certain block-size



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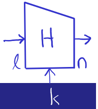

Definition 2 ($\ell(n)$-compression function)

A keyed function (family) $\{H : \mathcal{K} \times \{0,1\}^{\ell(n)} \to \{0,1\}^n\}$ is an $\ell(n)$-compression function if for every PPT collision-finder F, the following is negligible.

$$\Pr_{\substack{k \leftarrow \mathsf{Gen}(1^n) \\ (x_1 \overset{\neq}{,} x_2) \leftarrow \mathsf{F}(k)}} [H(k, x_1) = H(k, x_2)]$$

same length

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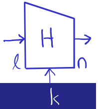

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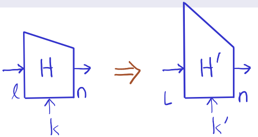
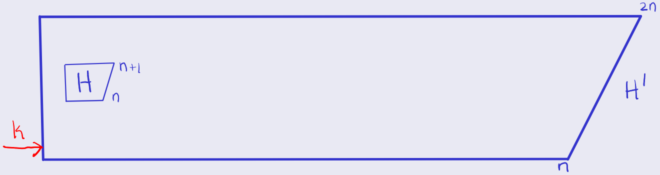
- Domain extension: $\ell(n)$-compression function $\Rightarrow L(n)$-compression function for $L(n) > \ell(n)$



Approach I: Merkle-Damgård Construction (Chaining)...

Construction 1 ($(n+1)$-compression fn. $H \Rightarrow 2n$-compression fn. H')

$\{0,1\}^{\ell(n)} \ni x = \boxed{x_1 \mid x_2 \mid x_3 \mid \cdots \mid x_i \mid \cdots \mid x_{2n-1} \mid x_{2n}}$



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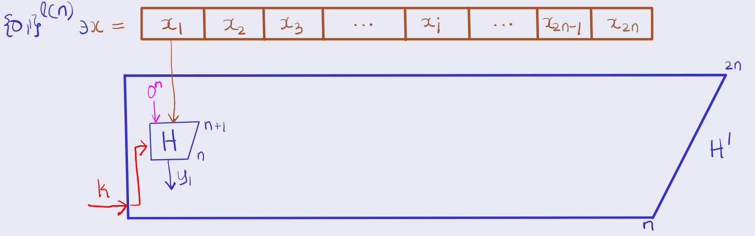

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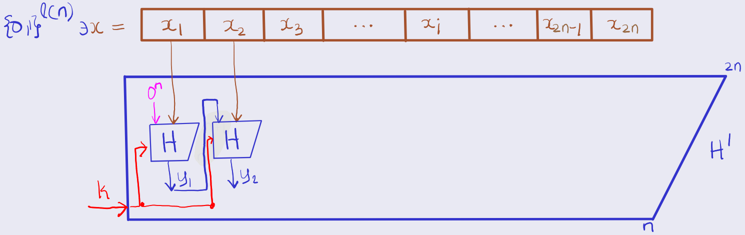

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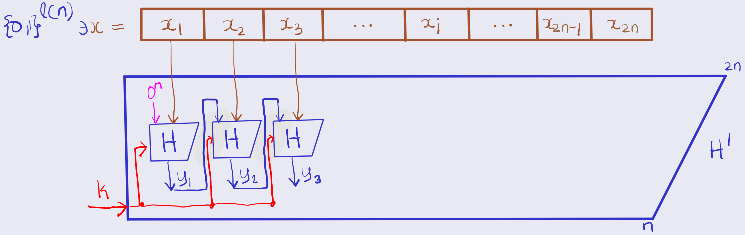

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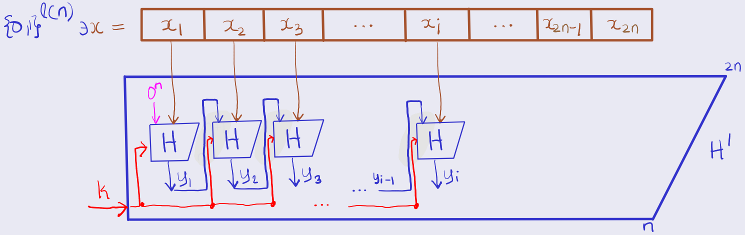

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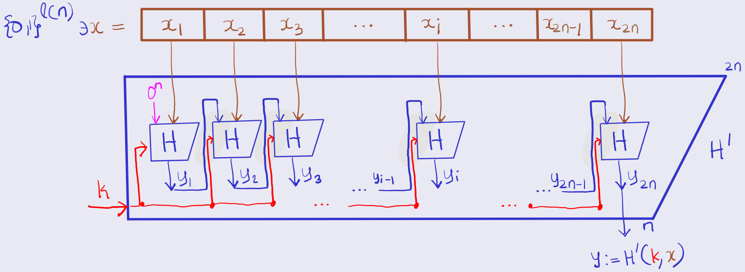

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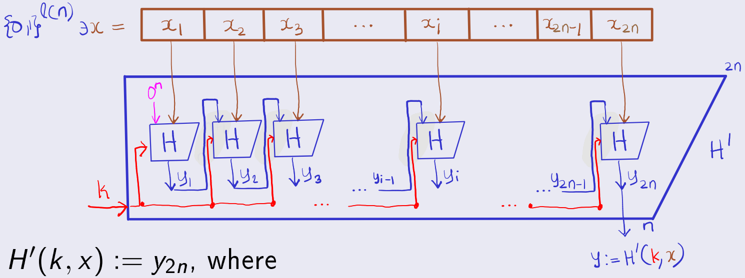

- $H'(k, x) := y_{2n}$, where
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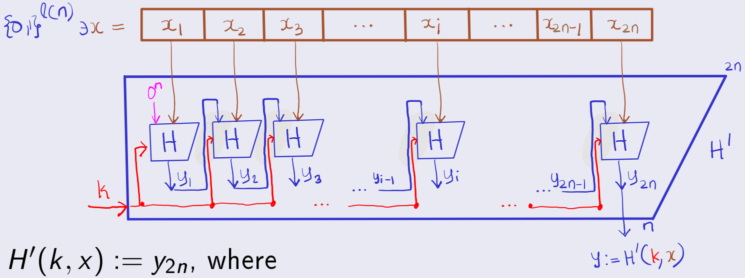

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Show that if H is a compression function then so is H'

- Is H' parallelisable?
- Can parts of input can be locally verified?

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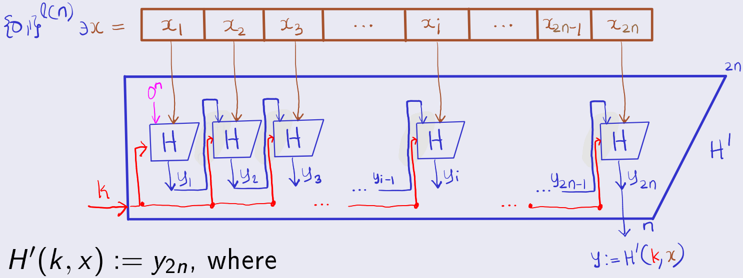

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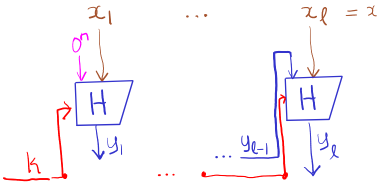

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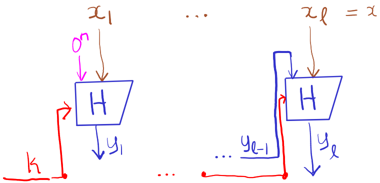

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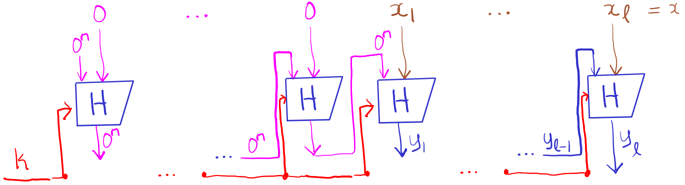

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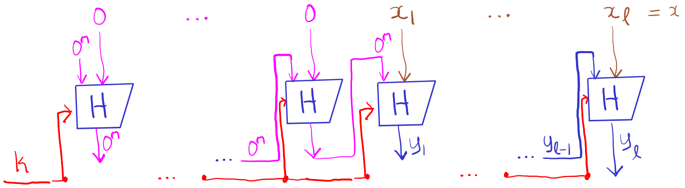


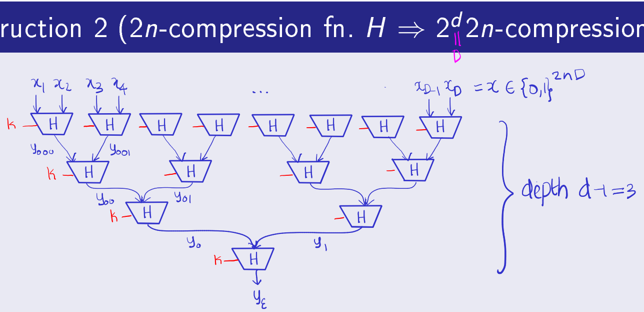
Exercise 4

Tweak Construction 1 to obtain CRHF (i.e., for domain $\{0,1\}^*$)

- Hint: add appropriate padding in the end

Approach II: Merkle's Construction (Tree-Based)

Construction 2 ($2n$-compression fn. $H \Rightarrow 2^d 2n$-compression fn. H')



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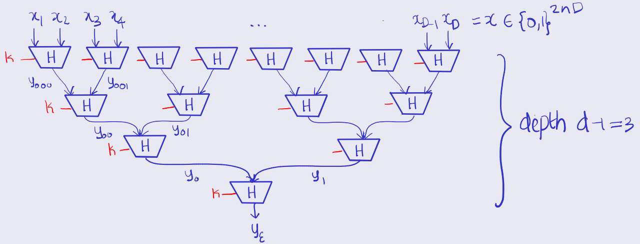
$\overset{\|}{D}$



- $\forall v \in \{0,1\}^d : y_v := x_v$ and $\forall v \in \{0,1\}^{<d} : y_v := H(k, y_{v\|0} \| y_{v\|1})$

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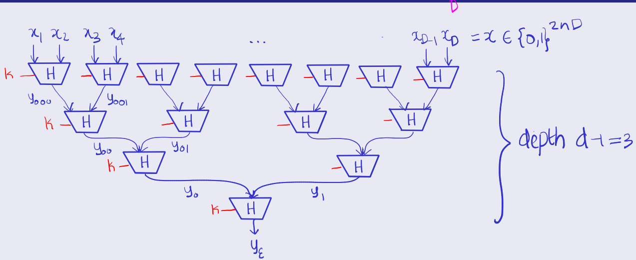

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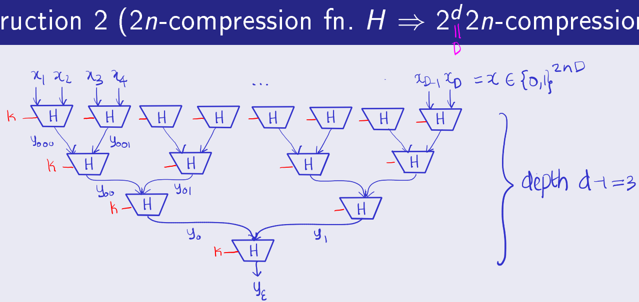

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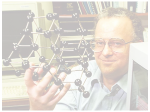
- Task: sign arbitrarily long messages
- Threat model: EU-CMA

Hash Function



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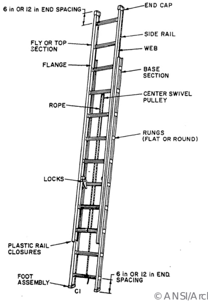
Domain Extension



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Old tricks: chain, tree-based constructions

How to Construct Compression Functions in Practice?

- *Unkeyed* compression fn. for fixed input (block)/output length

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```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab58712467eab 4004583eb8fb7f89
55ad340609f4b302 83e488832571415a 085125e8f7cdc99f d91dbdf280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e2b487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080a80d1e c69821bcb6a88393 96f9652b6ff72a70
```

```
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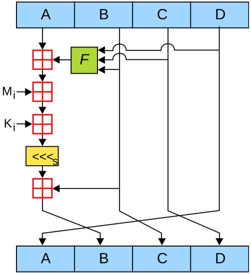
55ad340609f4b302 83e4888325f1415a 085125e8f7cdc99f d91dbd7280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e23487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080280d1e c69821bcb6a88393 96f965ab6ff72a70
```



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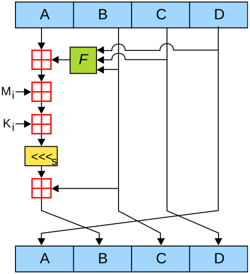
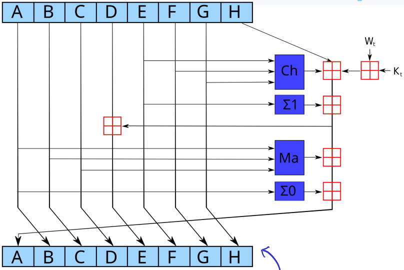
d8823e3156348f5b ae6dacd436c919c6 dd53e2b487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080a80d1e c69821bcb6a88393 96f9652b6ff72a70
```

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab50712467eab 4004583eb8fb7f89
55ad340609f4b302 83e4888325f1415a 085125e8f7cdc99f d91dbd7280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e23487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080280d1e c69821bcb6a88393 96f965ab6ff72a70
```





  - Secure-Hashing Algorithm (SHA) family
    - SHA2 (512/256,1024/512...): Davis-Meyer compression function
    - SHA3 (1152/224,576,512): “Sponge”-based compression function

How to Construct Compression Functions in Theory?

- Based on DLP in $\mathbb{Z}_p^\times$: $\{H : (\mathbb{Z}_p^\times)^2 \times \mathbb{Z}_p^2 \to \mathbb{Z}_p^\times\}$, where

$$H((g,h),(a,b)) := g^a h^b \bmod p$$

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  ? When is H compressing? When we set $p < 2^n$

  ? How to solve subset-sum given a collision?

Recap/Next Lecture

- Introduced a new primitive: collision-resistant hash function
  - Application: sign long messages
  - Also yields MAC for long messages! Refer to “HMAC”

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- Next lecture:
  - Efficient many-time signatures
  - New primitive: trap-door (one-way) permutation (TDP)
  - Proof in random oracle model (ROM)

References

1. You can read about hash functions and collision resistance in [KL14, Chapter 6].
2. Hash functions were first studied in [WC81], but they considered pairwise-independence/universal hashing
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