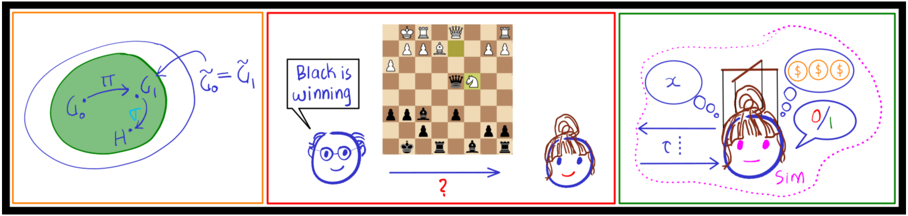
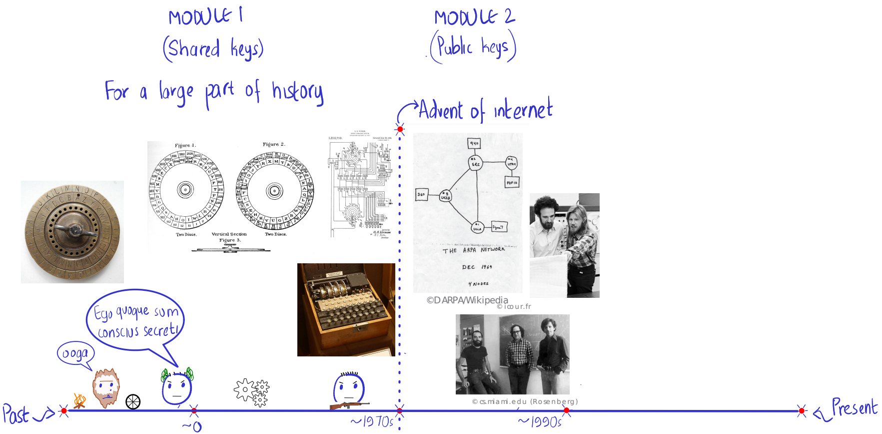


CS409m: Introduction to Cryptography

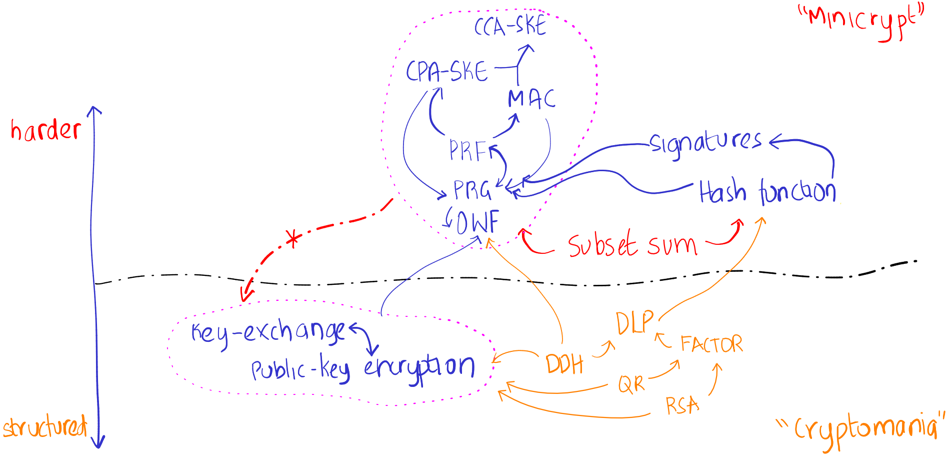
Lecture 18 (17/Oct/25)

Instructor: Chethan Kamath

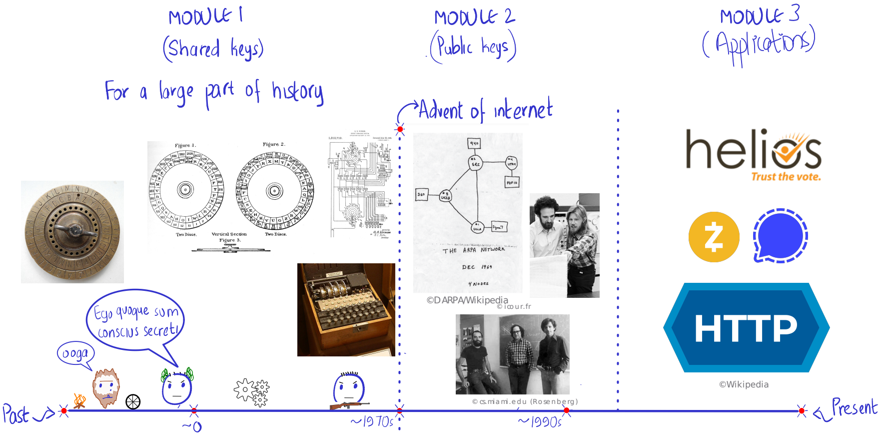
Journey So Far . . .



The Cryptographic Landscape



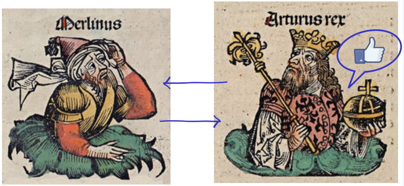
Plan for Module III: Applications!



Plan for Today's Lecture...

What really constitutes a proof?

Interactive Proof (IP)



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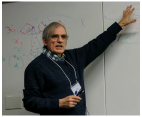

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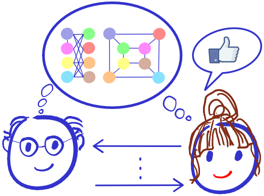
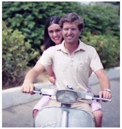
Zero-Knowledge IP





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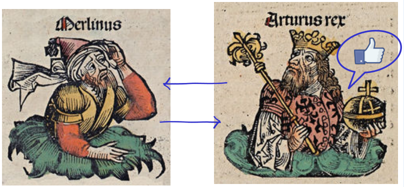

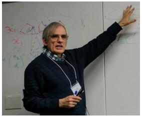

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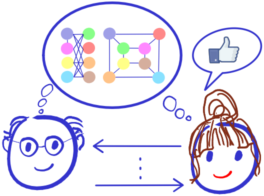
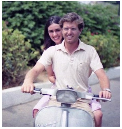
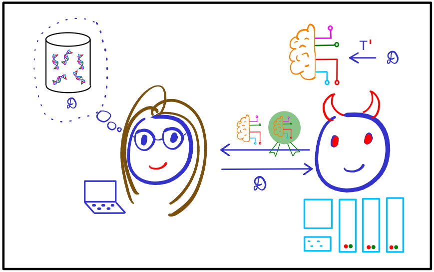




Main tools: simulation paradigm, Chernoff bound...

(ZK)IPs are Useful!

- Applications of IP: Verifiable outsourcing



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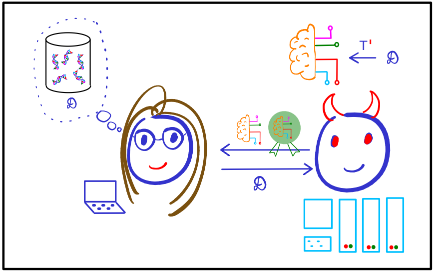
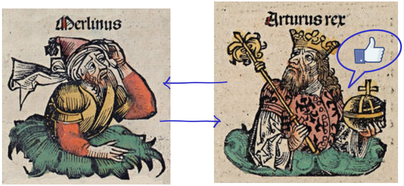


- Applications of ZKP:
  - eVoting: coming up in Lecture 20!
  - Crypto(currencies): prove validity of transaction without revealing details: coming up in Lecture 23!
  - Efficient digital signatures: Schnorr ID protocol

Plan for Today's Lecture...

What really constitutes a proof?

Interactive Proof (IP)



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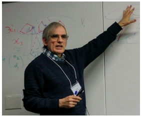

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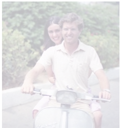
Zero-Knowledge IP



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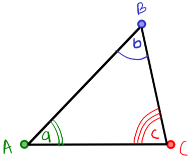
Main tools: simulation paradigm, Chernoff bound...

Traditional "NP" Proof...

- Axioms $\xrightarrow{\text{derivation rules}}$ theorems=true statements
  - E.g.: Axioms of Euclidean geometry

Theorem: "Sum of angles of a triangle equals 180°"



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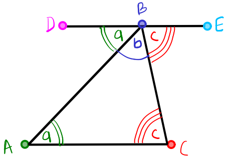

- Prover vs. verifier


Prover does the heavy lifting: derives the proof

1. Construct a line through B parallel to $\overline{AC}$
2. $\angle DBA = \angle a$ and $\angle EBC = \angle c$ (alternate interior angles)
3. $2 \Rightarrow \angle a + \angle b + \angle c = \angle DBA + \angle b + \angle EBC = 180°$

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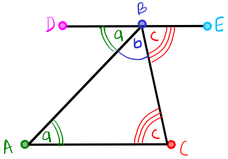

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  Verifier checks the proof, step by step

Traditional "NP" Proof...

- Corresponds to the *class* NP
  - A language $\mathcal{L} \in$ NP if there exists a polynomial-time *deterministic* machine V such that

statement

witness/proof

$$\forall x \in \mathcal{L}\ \exists \text{ ``short'' } \pi : \mathsf{V}(x, \pi) = 1$$

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  - NP is the *class* of all such $\mathcal{L}$s (e.g., Sudoku)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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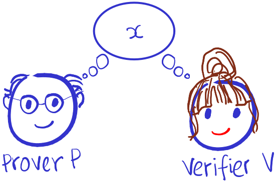

- "Proof system" view of NP
  - Prover P is *unbounded*: finds short proof π for x (if one exists)
  - Verifier V is *efficient*: checks proof π against the statement x

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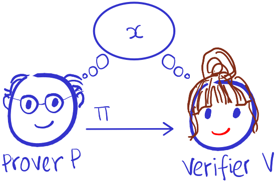

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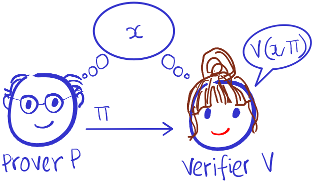

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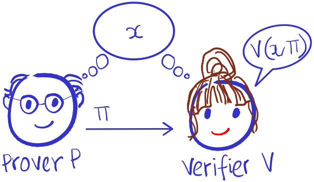

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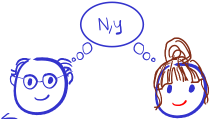
- "Proof system" view of NP
  - Prover P is *unbounded*: finds short proof π for x (if one exists)
  - Verifier V is *efficient*: checks proof π against the statement x
  - Completeness: $x \in \mathcal{L} \Rightarrow$ P finds $\pi \Rightarrow \mathrm{V}(x, \pi) = 1$
  - Soundness: $x \notin \mathcal{L} \Rightarrow \nexists$ "short" π s.t. $\mathrm{V}(x, \pi) = 1$

Which Languages have "NP" Proofs?

Quadratic residuosity (QR)



pq

$\mathcal{L}_{QR} = \{(N, y) : \exists x \in \mathbb{Z}_N^{\times} \text{ s.t. } y = x^2 \bmod N\}$

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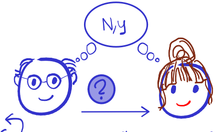

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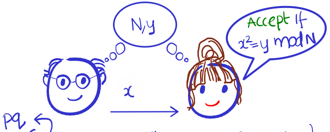

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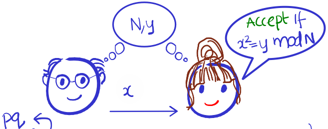

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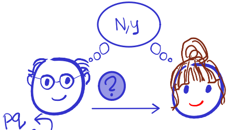
Quad. non-residuosity (QNR)



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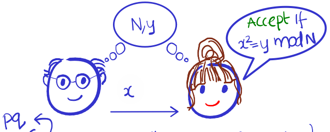

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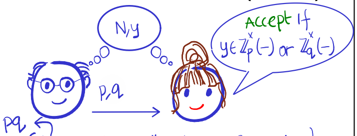

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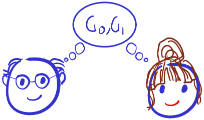
Graph isomorphism (GI)



$\mathcal{L}_{GI}=\{(G_0,G_1): \exists \text{permutation } \pi \text{ s.t. } G_1=\pi(G_0)\}$

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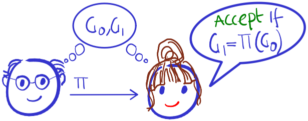

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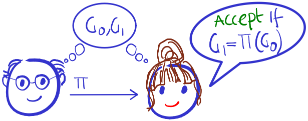

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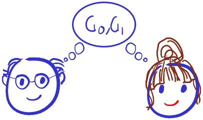

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p,q

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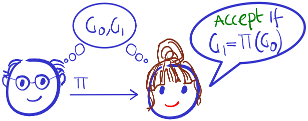

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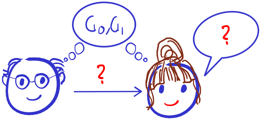

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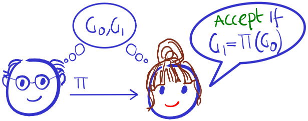

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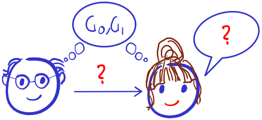
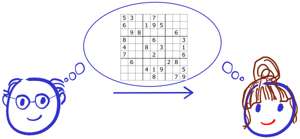


$\mathcal{L}_{GNI}=\{(G_0,G_1) : \nexists \text{permutation } \pi \text{ s.t. } G_1=\pi(G_0)\}$

Sudoku



Chess

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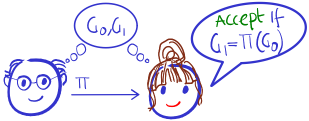

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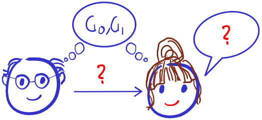

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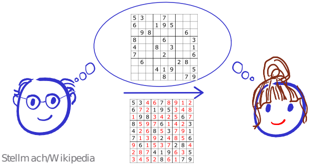
Sudoku



Chess

Which Languages have “NP” Proofs?

Quadratic residuosity (QR)

$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$

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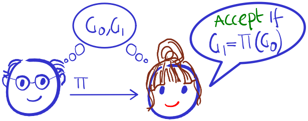

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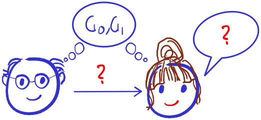

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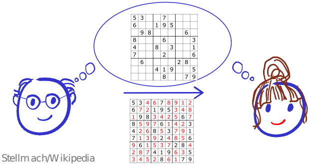

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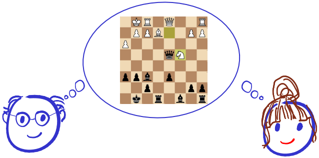

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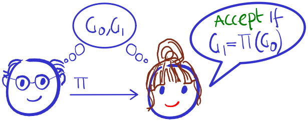

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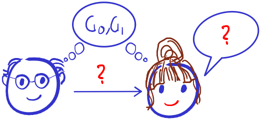

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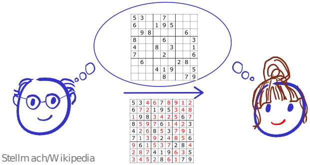

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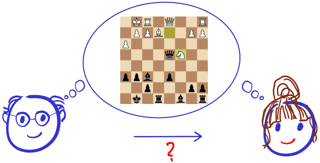
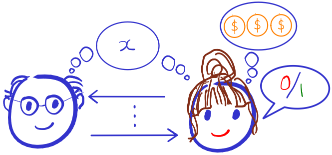


Interactive Proof (IP)

- Difference from NP proofs:
  1. Verifier V is *randomised*
  2. Prover P and V *interact* and V accepts/rejects in the end



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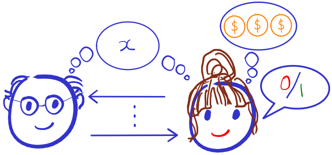

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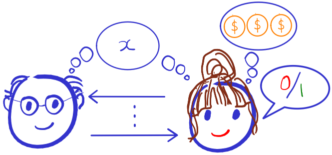

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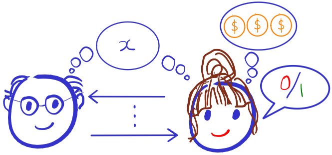

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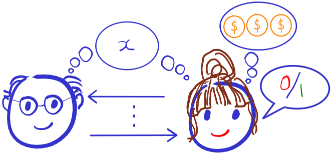

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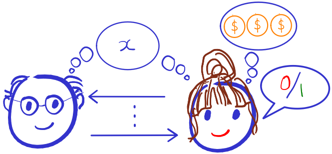

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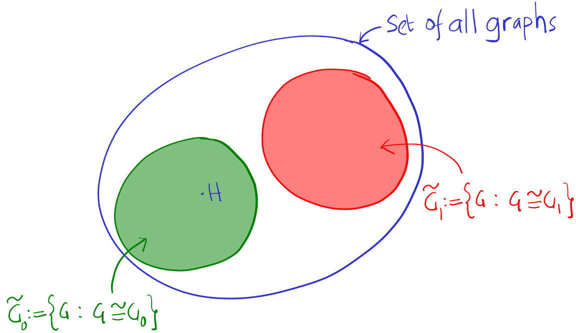
Exercise 1 (Definition 1 is robust)

Show that languages captured by Definition 1 doesn't change when $\epsilon_c \leq 1/2^{|x|}$ and $\epsilon_s \leq 1/2^{|x|}$ (Hint: repeat protocol, use Chernoff bound)

Idea: $G_0 \ncong G_1 \Rightarrow$ for any graph H, *$G_0 \cong H$ and $G_1 \cong H$ both cannot hold*



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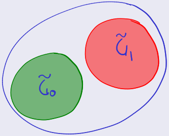
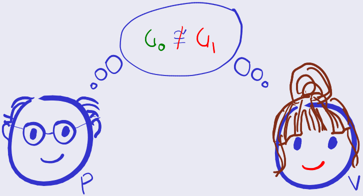
Protocol 1 (Π_{GNI}: IP for GNI)





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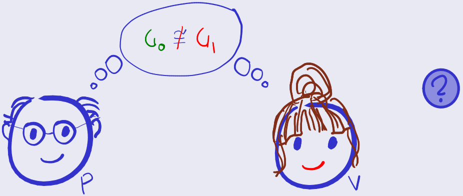

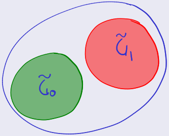

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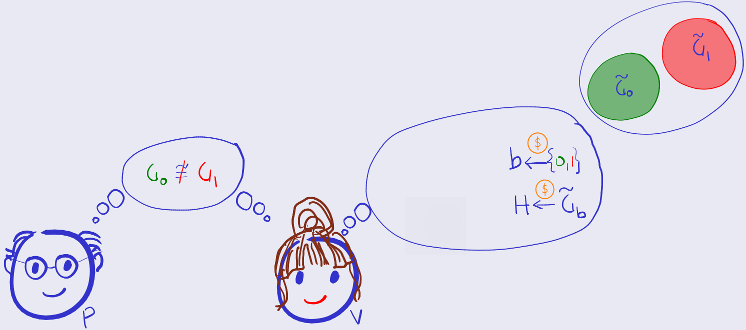

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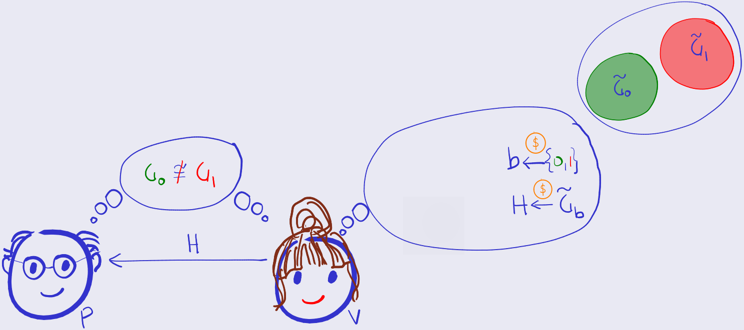

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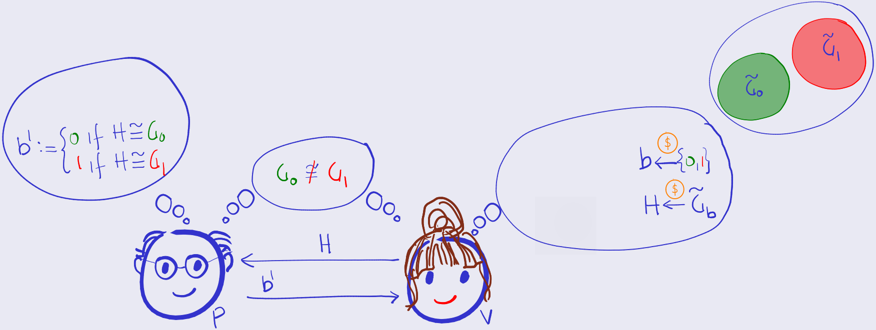

Power of Randomness+Interaction: IP for GNI...

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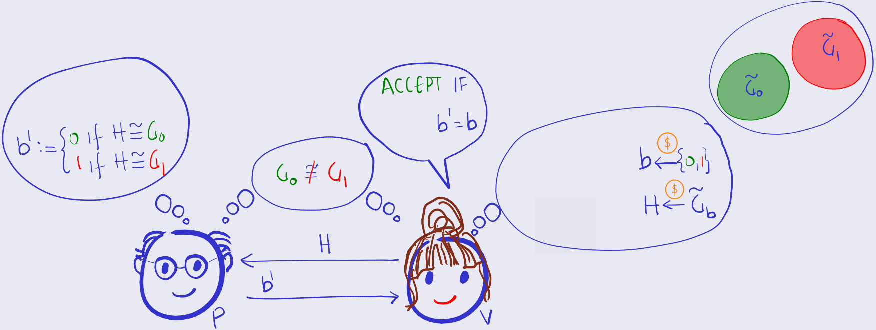

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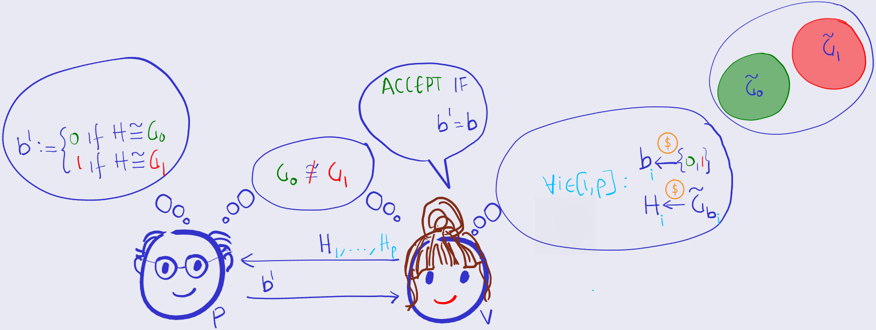

Parallel/sequentially repeat to boost soundness

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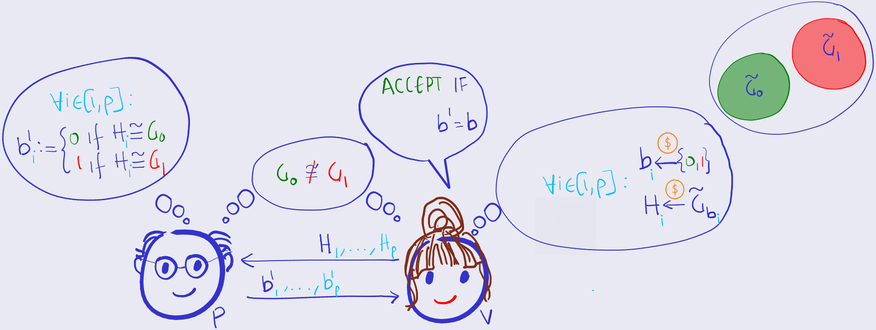

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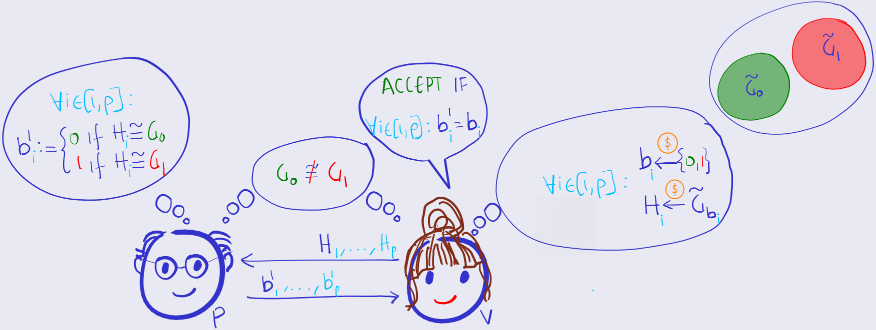

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Theorem 1

Π_{GNI} *is an IP for* $\mathcal{L}_{GNI}$

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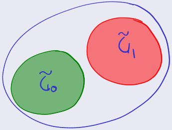

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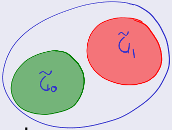

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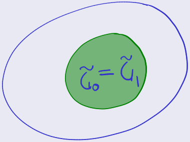
- Soundness:
  - $G_0 \cong G_1 \Rightarrow H_i$ loses information about bits b_i
  - Hence best P* can do is guess b_is



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□

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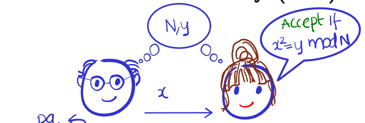

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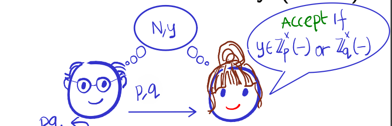

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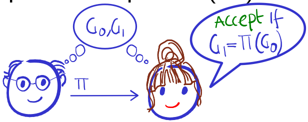

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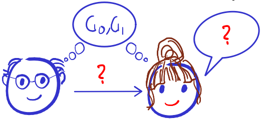

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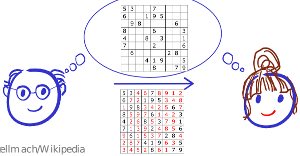
Sudoku



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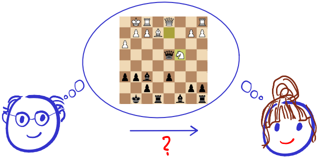

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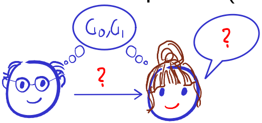

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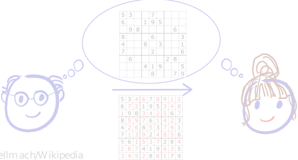

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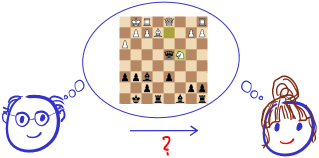

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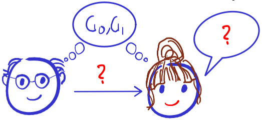

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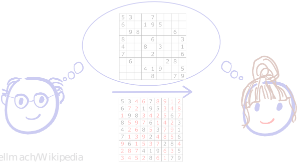

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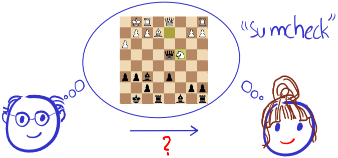

Which Languages have IPs? PSPACE Languages

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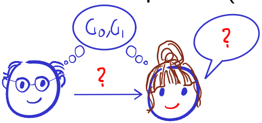

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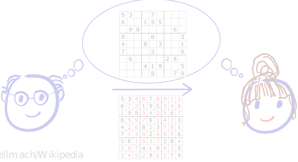
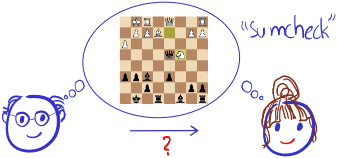
Sudoku



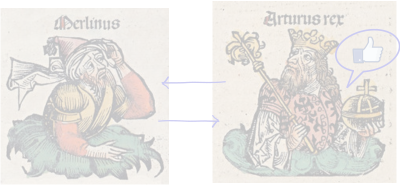
Chess



Plan for Today's Lecture...

What really constitutes a proof?

NEW Interactive Proof (IP)



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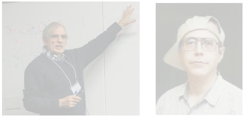

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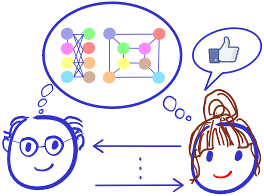
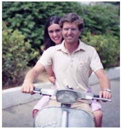
Zero-Knowledge IP NEW



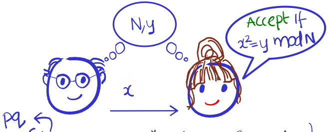


© Oded Goldreich

Main tools: simulation paradigm, Chernoff bound...

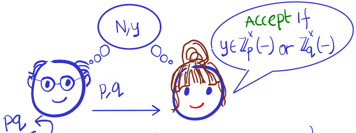
Any Issues with the NP Proofs We Saw?

Quadratic residuosity (QR)



$$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$$

Quad. non-residuosity (QNR)



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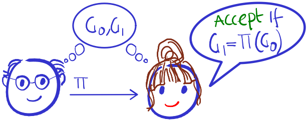

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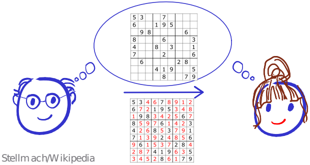

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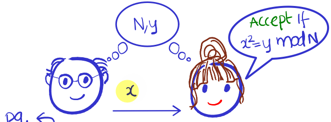

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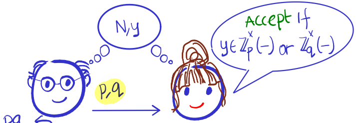

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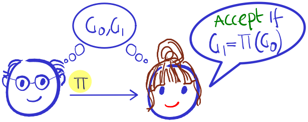

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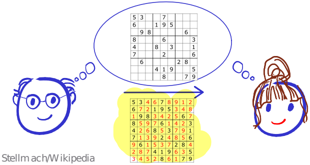
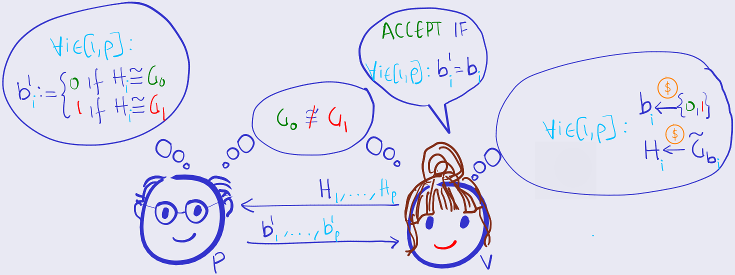


- Verifier gains "non-trivial knowledge" about witness w
  - Not desirable, e.g., when $x = pk$ and $w = sk$ (identification)

What About the IP We Saw?

Protocol 1 (Π_{GNI}: IP for GNI)



Parallel/sequentially repeat to boost soundness

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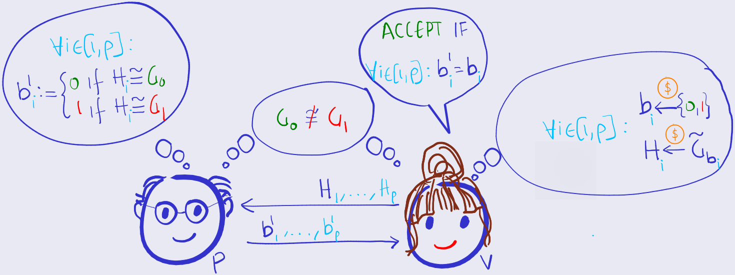

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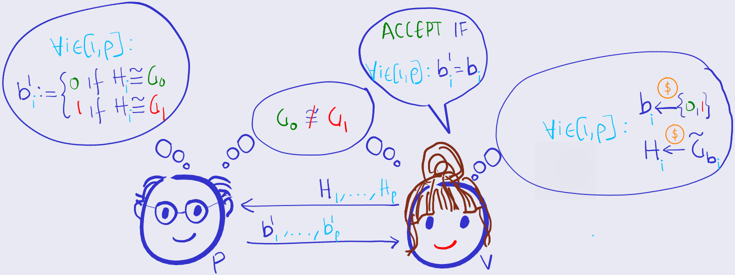

Parallel/sequentially repeat to boost soundness

- Seems V *gains no knowledge* beyond validity of the statement
- We will see that Π_{GNI} is (honest-verifier) zero-knowledge!

How to Capture "V Gains No Knowledge"?

- Knowledge vs. information in the information-theoretic sense
  - Knowledge is computational

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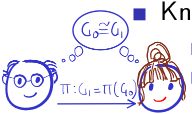
    - Given (G_0, G_1), the isomorphism π contains no *information*
    - But when given π, V "gains knowledge" since she couldn't have computed π herself



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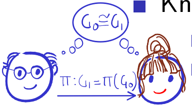

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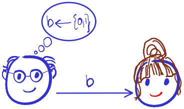

    - Flipping a *private* fair coin b and (later) revealing its outcome leads to V gaining *information*
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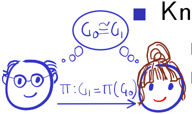

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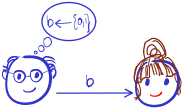
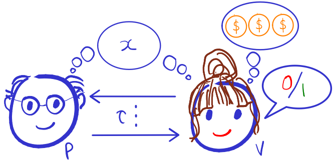


    - Flipping a *private* fair coin b and (later) revealing its outcome leads to V gaining *information*
    - But V *does not gain knowledge*: she could herself have tossed the private coin and revealed it

Intuitively, "V gains no knowledge" if anything (other than the validity of x) V can *compute* after the interaction, V could have computed *without it*

Defining Zero Knowledge via Simulators...

- Formalised via "simulation paradigm": $\mathit{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V}\rangle(x))$ can be *efficiently* simulated given only the instance



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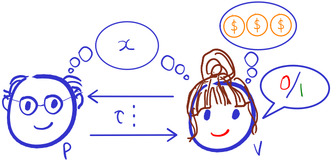

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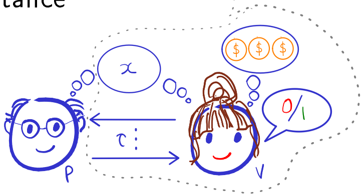

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An IP Π is *honest-verifier perfect* ZK if there exists a PPT *simulator* Sim such that for *all* distinguishers D and all $x \in \mathcal{L}$, the following is zero

$$\Pr[\mathsf{D}(\mathit{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle (x))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(x)) = 1]$$

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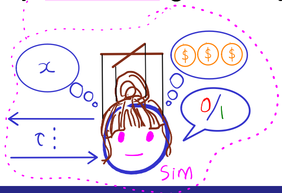

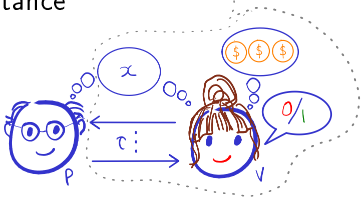

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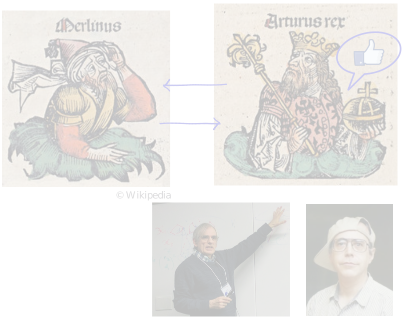
Exercise 2

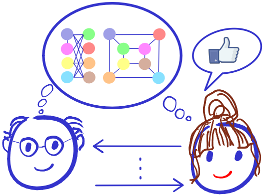
What happens when one invokes the simulator on $x \notin \mathcal{L}$?

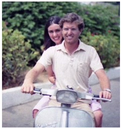
What really constitutes a proof?

Interactive Proof (IP)



Zero-Knowledge IP



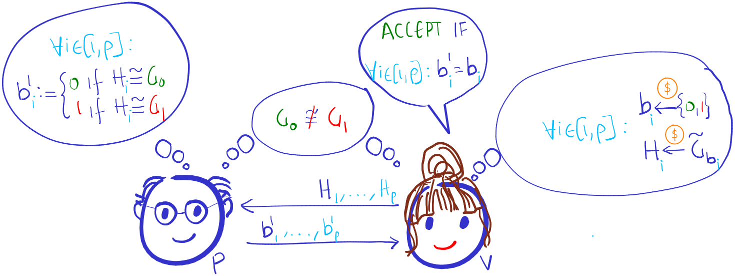


Main tools: simulation paradigm, Chernoff bound...

Π_{GNI} is *Honest-Verifier* ZK

Theorem 2

Π_{GNI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GNI}$



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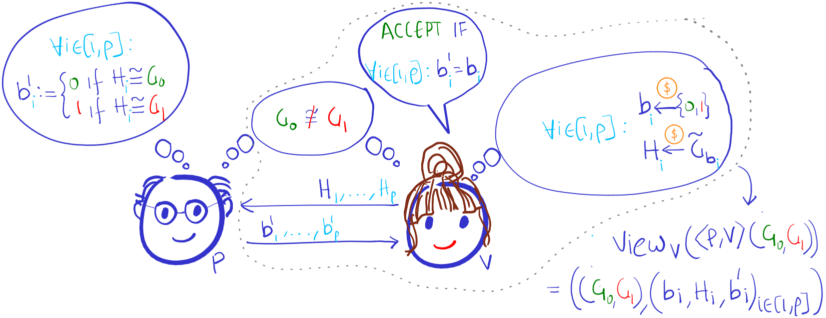

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Proof.

$$\mathrm{View}_V(\langle P,V\rangle(G_0,G_1)) := \left((G_0,G_1),(b_i,H_i,b_i')_{i\in[1,\ell]}\right)$$

$\forall\, G_0 \not\cong G_1$:

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$\mathrm{Sim}(G_0,G_1) := \forall i\in[1,p]$: sample $b_i \leftarrow \{0,1\}$ and $H_i \leftarrow \tilde{G}_{b_i}$

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$\forall\, G_0 \not\cong G_1$: $\mathrm{View}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\mathrm{Sim}(G_0,G_1)$.

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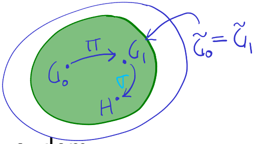
Exercise 3

1. What happens if V is "malicious" and can deviate from protocol?
2. Using ideas from Π_{GNI}, build honest-verifier ZKP for $\mathcal{L}_{QNR}$

Honest-Verifier ZKP for GI...



Idea for ZK:

1. $G_0 \overset{\pi}{\cong} G_1 \Rightarrow$ if $G_1 \overset{\sigma}{\cong} H$ then $G_0 \overset{\sigma\circ\pi}{\cong} H$
2. Prover sends a random H s.t. $G_1 \overset{\sigma}{\cong} H$
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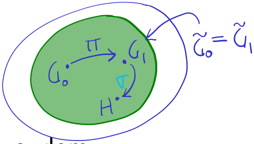
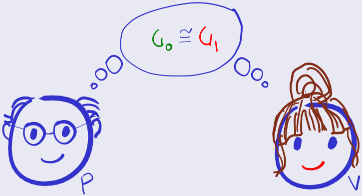
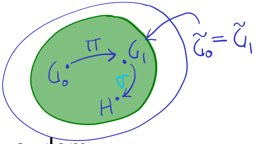


Protocol 2 (Π_{GI}: IP for GI)





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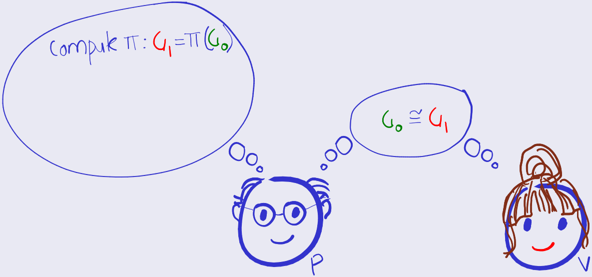

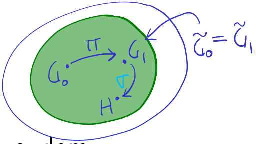

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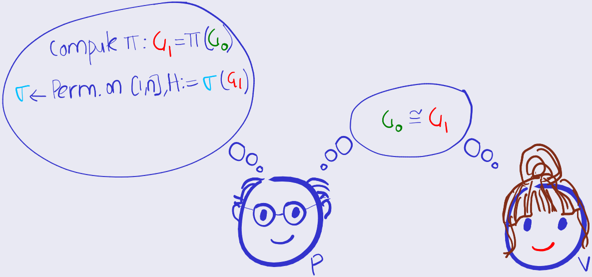

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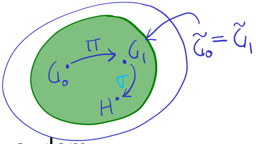

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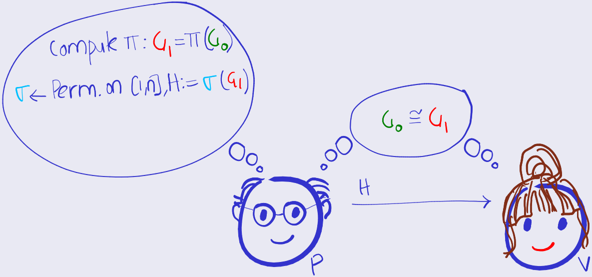

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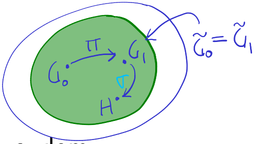
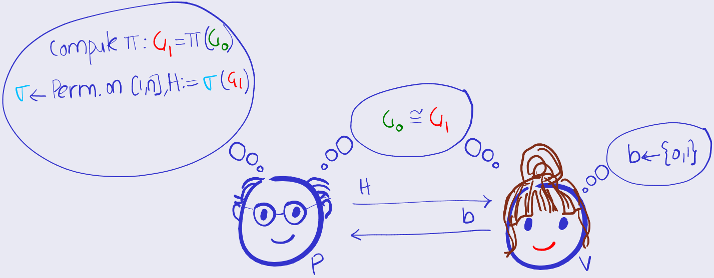
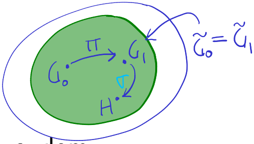


Protocol 2 (Π_{GI}: IP for GI)



Honest-Verifier ZKP for GI...



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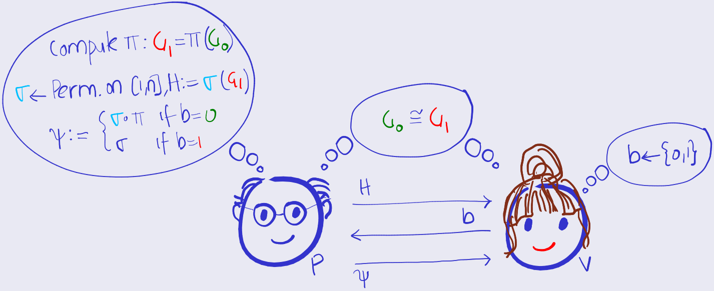

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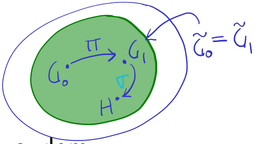

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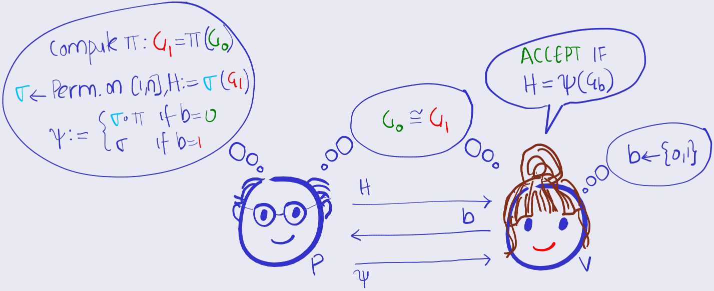

Honest-Verifier ZKP for GI

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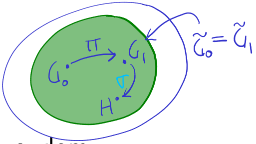

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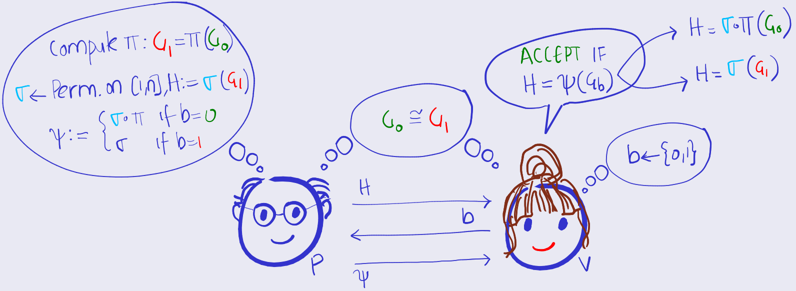

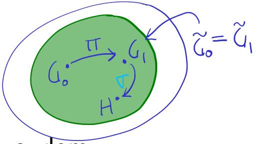

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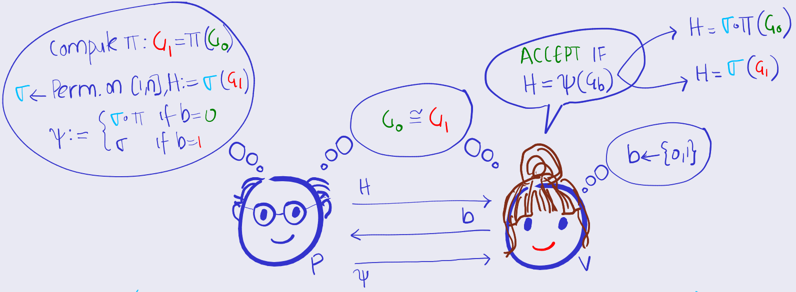

(Parallel/sequentially repeat to boost soundness)

Honest-Verifier ZKP for GI...

Theorem 3

Π_{GI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GI}$

Honest-Verifier ZKP for GI...

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Proof (idea for ZK: out of order sampling).

- Completeness: $G_0 \cong G_1 \Rightarrow$ P can answer both challenges $\Rightarrow$ V always accepts
- Soundness: $G_0 \ncong G_1 \Rightarrow$ for *any* H P* commits to, $G_0 \cong H$ *and* $G_1 \cong H$ *cannot both hold* $\Rightarrow$ best P* can do is guess b

Honest-Verifier ZKP for GI...

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$$\mathrm{View}_V(\langle P,V\rangle(G_0,G_1)) := ((G_0,G_1),(H,b,\psi))$$

$\forall\, G_0 \overset{\pi}{\cong} G_1:$

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$\forall\, G_0 \overset{\pi}{\cong} G_1$: $\mathrm{Sim}(G_0,G_1)$:

Honest-Verifier ZKP for GI...

Theorem 3

Π_{GI} *is honest-verifier perfect zero-knowledge IP for* $\mathcal{L}_{GI}$

Proof (idea for ZK: out of order sampling).

- Completeness: $G_0 \cong G_1 \Rightarrow$ P can answer both challenges $\Rightarrow$ V always accepts
- Soundness: $G_0 \not\cong G_1 \Rightarrow$ for *any* H P* commits to, $G_0 \cong H$ *and* $G_1 \cong H$ *cannot both hold* $\Rightarrow$ best P* can do is guess b
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$$\mathrm{View}_V(\langle P,V\rangle(G_0,G_1)) := ((G_0,G_1),(H,b,\psi))$$

$\forall\, G_0 \overset{\pi}{\cong} G_1$: $\mathrm{Sim}(G_0,G_1)$: sample $b \leftarrow \{0,1\}$, $\psi \leftarrow$ permutation on $[1,n]$

set $H := \psi(G_b)$

o/p $((G_0,G_1),(H,b,\psi))$

$\forall\, G_0 \cong G_1$: $\mathrm{View}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\mathrm{Sim}(G_0,G_1)$. □

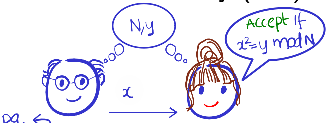
Honest-Verifier ZKP for GI...

Exercise 4

1. What happens if V is "malicious" and can deviate from protocol?
2. Using ideas from Π_{GI}, build honest-verifier ZKP for $\mathcal{L}_{QR}$

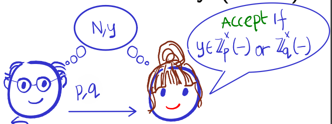
Which Languages have ZKPs?

Quadratic residuosity (QR)



$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$

Quad. non-residuosity (QNR)



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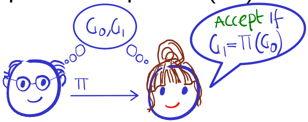
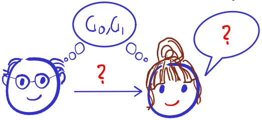
Graph isomorphism (GI)



$\mathcal{L}_{GI} = \{(G_0,G_1) : \exists \text{ permutation } \pi \text{ s.t. } G_1 = \pi(G_0)\}$

Graph non-isomorphism (GNI)



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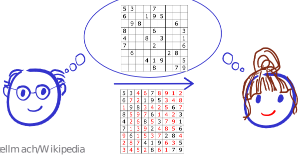
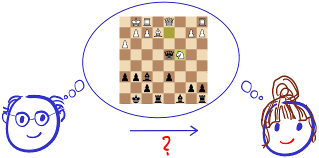
Sudoku



Chess



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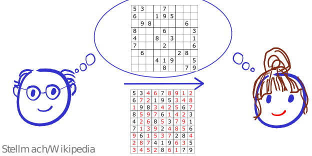
L19

Sudoku



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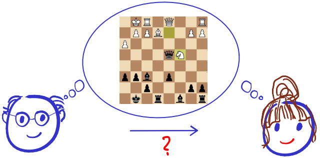

Which Languages have ZKPs? PSPACE Languages

Quadratic residuosity (QR)

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L19

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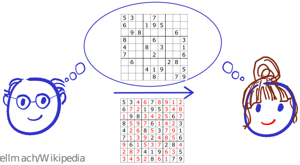

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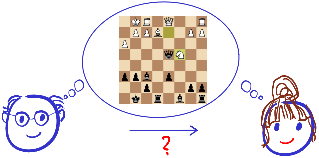

Are Randomness and Interaction Necessary?

$\leftrightarrows$ Interaction is necessary

Fact 4

If $\mathcal{L}$ has a non-interactive (i.e, one-message) ZKP then $\mathcal{L}$ is "trivial"

Are Randomness and Interaction Necessary?

Interaction is necessary

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Randomness is necessary

Exercise 5

If $\mathcal{L}$ has an IP with deterministic verifier then $\mathcal{L} \in \mathrm{NP}$

Fact 5

If $\mathcal{L}$ has an ZKP with deterministic verifier then $\mathcal{L}$ is "trivial"

Recap/Next Lecture

- Traditional “NP” proofs vs *interactive* proofs
    - IP is more powerful: IP for GNI

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- Next Lecture:
  - Computational ZKP for all of NP!
    - New cryptographic primitive: commitment schemes
  - Return to ID protocols: zero-knowledge proof of knowledge

References

1. [Gol01, Chapter 4] for details of today's lecture
2. [GMR89] for definitional and philosophical discussion on ZK
3. The ZKPs for GI and GNI are taken from [GMR89, GMW91]
4. IP for all of PSPACE is due to [LFKN92, Sha90]. Computational ZKP for all of PSPACE is due to [GMW91].

Shafi Goldwasser, Silvio Micali, and Charles Rackoff.
The knowledge complexity of interactive proof systems.
SIAM J. Comput., 18(1):186–208, 1989.

Oded Goldreich, Silvio Micali, and Avi Wigderson.
Proofs that yield nothing but their validity for all languages in NP have zero-knowledge proof systems.
J. ACM, 38(3):691–729, 1991.

Oded Goldreich.
The Foundations of Cryptography - Volume 1: Basic Techniques.
Cambridge University Press, 2001.

Carsten Lund, Lance Fortnow, Howard Karloff, and Noam Nisan.
Algebraic methods for interactive proof systems.
J. ACM, 39(4):859–868, October 1992.

Adi Shamir.
IP=PSPACE.
In *31st FOCS*, pages 11–15. IEEE Computer Society Press, October 1990.