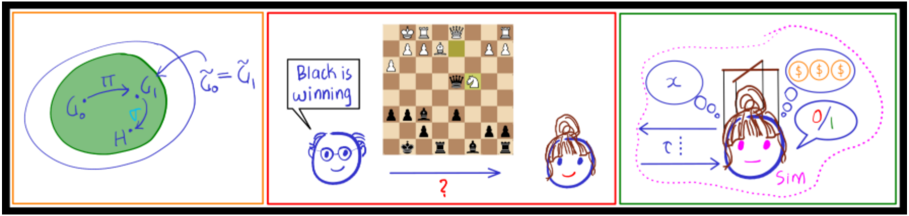


CS409m: Introduction to Cryptography

Lecture 18 (17/Oct/25)

Instructor: Chethan Kamath

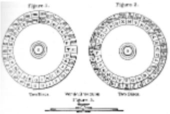
Journey So Far...

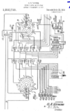
MODULE 1
(Shared keys)

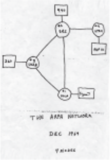
MODULE 2
(Public keys)

For a large part of history

Advent of internet







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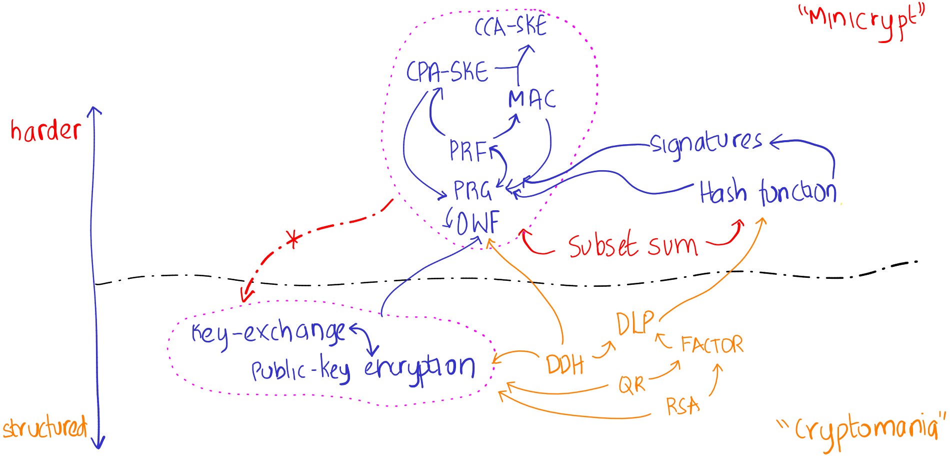
Ego quoque sum conscius secreti

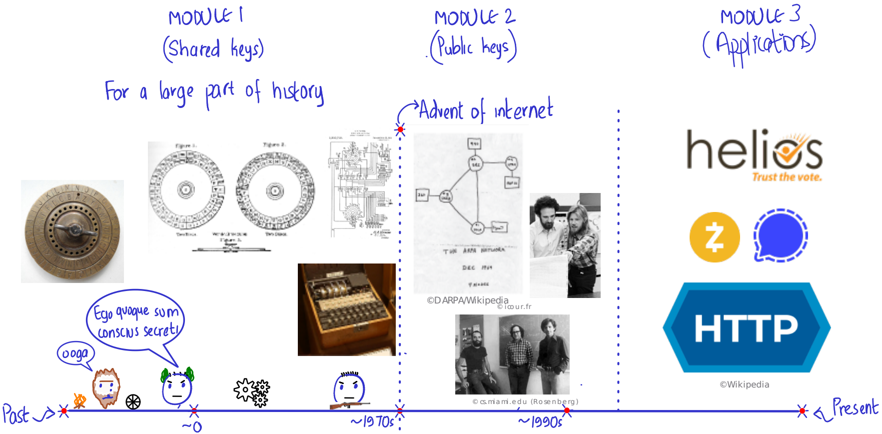
ooga

Past → ~0 ~1970s ~1990s ← Present

The Cryptographic Landscape



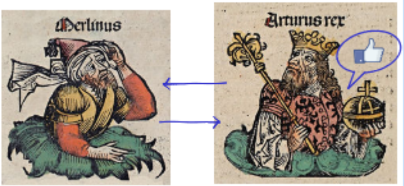
Plan for Module III: Applications!



Plan for Today's Lecture...

What really constitutes a proof?

Interactive Proof (IP)



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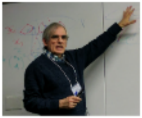

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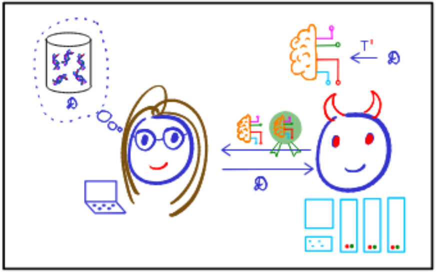
Zero-Knowledge IP

© Oded Goldreich

Main tools: simulation paradigm, Chernoff bound...

(ZK)IPs are Useful!

- Applications of IP: Verifiable outsourcing



- Applications of ZKP:
  - eVoting: coming up in Lecture 20!
  - Crypto(currencies): prove validity of transaction without revealing details: coming up in Lecture 23!
  - Efficient digital signatures: Schnorr ID protocol

Plan for Today's Lecture...

What really constitutes a proof?

NEW **Interactive Proof (IP)**

Zero-Knowledge IP NEW

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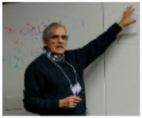

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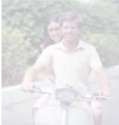

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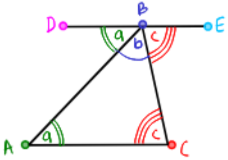
Main tools: simulation paradigm, Chernoff bound...

Traditional "NP" Proof

- Axioms $\xrightarrow{\text{derivation rules}}$ theorems=true statements
  - E.g.: Axioms of Euclidean geometry

    ↳Theorem: "Sum of angles of a triangle equals 180°"



- Prover vs. verifier

  Prover does the heavy lifting: derives the proof

  1. Construct a line through B parallel to $\overline{AC}$
  2. $\angle DBA = \angle a$ and $\angle EBC = \angle c$ (alternate interior angles)
  3. $2 \Rightarrow \angle a + \angle b + \angle c = \angle DBA + \angle b + \angle EBC = 180°$

  Verifier checks the proof, step by step

Traditional "NP" Proof...

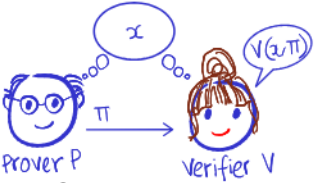
- Corresponds to the *class* NP
  - A language $\mathcal{L} \in \mathrm{NP}$ if there exists a polynomial-time *deterministic* machine V such that

witness/proof

statement $\quad \forall x \in \mathcal{L}\ \exists\ \text{"short"}\ \pi : \mathrm{V}(x, \pi) = 1$

  - NP is the *class* of all such $\mathcal{L}$s (e.g., Sudoku)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9



- "Proof system" view of NP
  - Prover P is *unbounded*: finds short proof π for x (if one exists)
  - Verifier V is *efficient*: checks proof π against the statement x
  - Completeness: $x \in \mathcal{L} \Rightarrow$ P finds $\pi \Rightarrow \mathrm{V}(x, \pi) = 1$
  - Soundness: $x \notin \mathcal{L} \Rightarrow \nexists$ "short" π s.t. $\mathrm{V}(x, \pi) = 1$

Which Languages have "NP" Proofs?

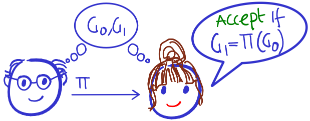
Quadratic residuosity (QR)

$$\mathcal{L}_{QR}=\{(N,y):\exists x\in\mathbb{Z}_N^\times \text{ s.t. } y=x^2 \bmod N\}$$

Quad. non-residuosity (QNR)

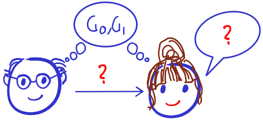
$$\mathcal{L}_{QNR}=\{(N,y):\nexists x\in\mathbb{Z}_N^\times \text{ s.t. } y=x^2 \bmod N\}$$

Graph isomorphism (GI)



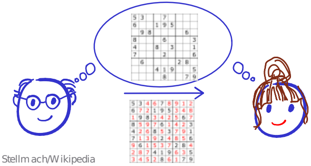
$$\mathcal{L}_{GI}=\{(G_0,G_1):\exists \text{permutation } \pi \text{ s.t. } G_1=\pi(G_0)\}$$

Graph non-isomorphism (GNI)

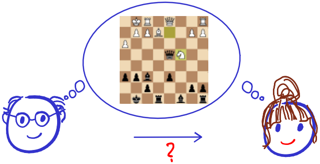


$$\mathcal{L}_{GNI}=\{(G_0,G_1):\nexists \text{permutation } \pi \text{ s.t. } G_1=\pi(G_0)\}$$

Sudoku



Chess



Interactive Proof (IP)

Difference from NP proofs:

1. Verifier V is *randomised*
2. Prover P and V *interact* and V accepts/rejects in the end

Definition 1

An interactive protocol (P, V) for a language $\mathcal{L}$ is an *interactive proof (IP)* system if the following holds:

completeness error $\varepsilon_c(n)$

- Completeness: for every $x \in \mathcal{L}$, $\Pr[1 \leftarrow \langle \mathrm{P}, \mathrm{V}\rangle(x)] \geq 1 - 1/3$
- Soundness: for every $x \notin \mathcal{L}$ and *malicious* prover P^*,

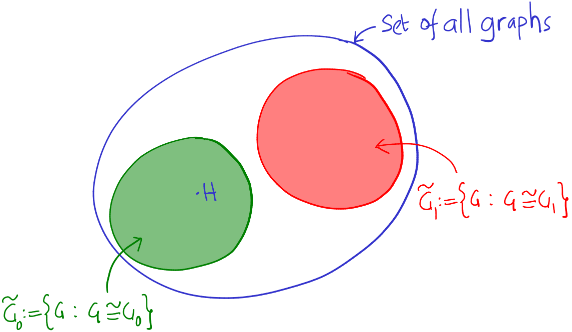
$$\Pr[1 \leftarrow \langle \mathrm{P}^*, \mathrm{V}\rangle(x)] \leq 1/3$$

soundness error $\varepsilon_s(n)$

Exercise 1 (Definition 1 is robust)

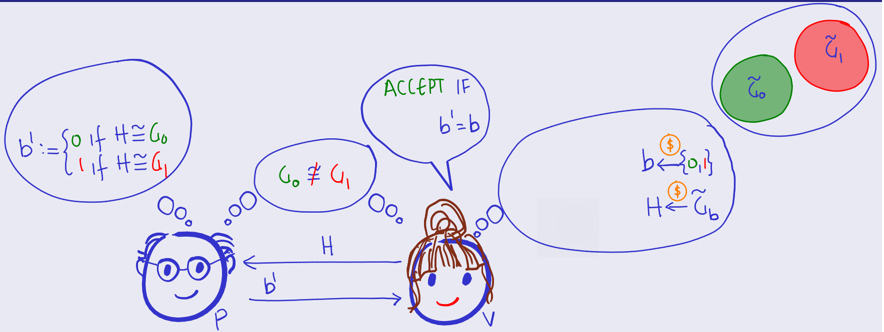
Show that languages captured by Definition 1 doesn't change when $\epsilon_c \leq 1/2^{|x|}$ and $\epsilon_s \leq 1/2^{|x|}$ (Hint: repeat protocol, use Chernoff bound)

Idea: $G_0 \not\cong G_1 \Rightarrow$ for any graph H, $G_0 \cong H$ *and* $G_1 \cong H$ *both cannot hold*



Idea: $G_0 \not\cong G_1 \Rightarrow$ for any graph H, *$G_0 \cong H$ and $G_1 \cong H$ both cannot hold*

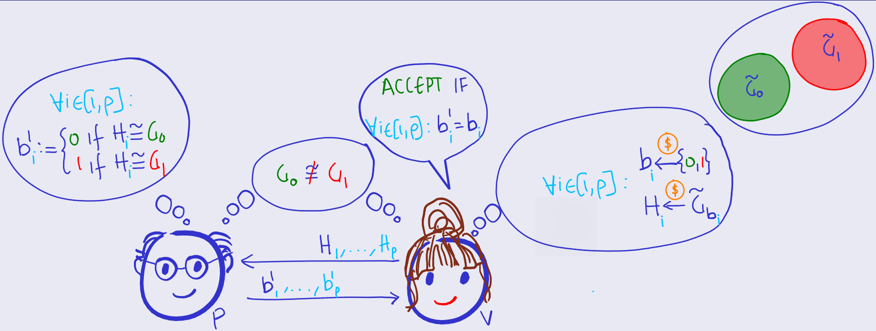
Protocol 1 (Π_{GNI}: IP for GNI)



Power of Randomness+Interaction: IP for GNI...

Idea: $G_0 \ncong G_1 \Rightarrow$ for any graph H, *$G_0 \cong H$ and $G_1 \cong H$ both cannot hold*

Protocol 1 (Π_{GNI}: IP for GNI)



Parallel/sequentially repeat to boost soundness

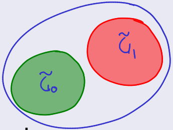
Power of Randomness+Interaction: IP for GNI...

Theorem 1

Π_{GNI} is an IP for $\mathcal{L}_{GNI}$

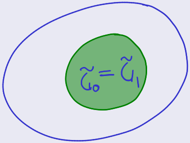
Proof.

- Completeness:
  - $G_0 \ncong G_1 \Rightarrow$ P can recover b_i from H_i with certainty



$$\Pr[1 \leftarrow \langle \mathsf{P}, \mathsf{V} \rangle (G_0, G_1)] = 1 \geq 2/3$$

- Soundness:
  - $G_0 \cong G_1 \Rightarrow H_i$ loses information about bits b_i
  - Hence best P* can do is guess b_is



$$\Pr[1 \leftarrow \langle \mathsf{P}^*, \mathsf{V} \rangle (G_0, G_1)] = 1/2^{\rho} < 1/3$$

□

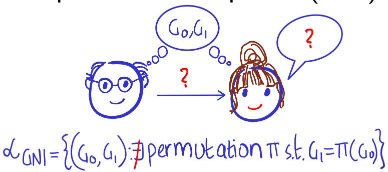
Which Languages have IPs? PSPACE Languages

Quadratic residuosity (QR)

Quad. non-residuosity (QNR)

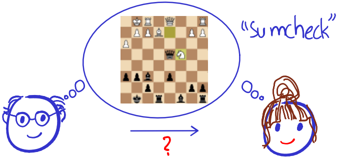
Graph isomorphism (GI)

Graph non-isomorphism (GNI)



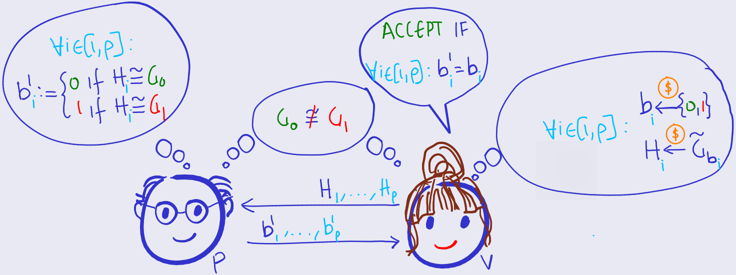
Sudoku

Chess



What About the IP We Saw?

Protocol 1 (Π_{GNI}: IP for GNI)

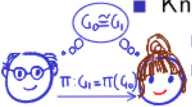


Parallel/sequentially repeat to boost soundness

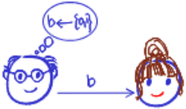
- Seems V *gains no knowledge* beyond validity of the statement
- We will see that Π_{GNI} is (honest-verifier) zero-knowledge!

How to Capture "V Gains No Knowledge"?

- Knowledge vs. information in the information-theoretic sense
  - Knowledge is computational: e.g., consider NP proof for GI



    - Given (G_0, G_1), the isomorphism π contains no *information*
    - But when given π, V "gains knowledge" since she couldn't have computed π herself
  - Knowledge pertains to public objects:



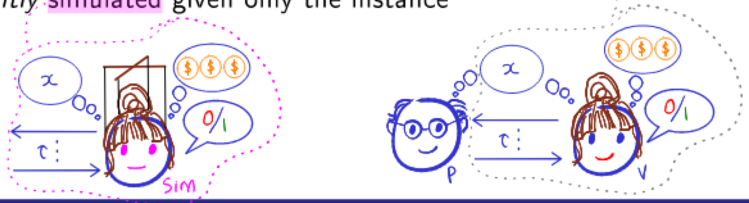
    - Flipping a *private* fair coin b and (later) revealing its outcome leads to V gaining *information*
    - But V *does not gain knowledge*: she could herself have tossed the private coin and revealed it

Intuitively, "V gains no knowledge" if anything (other than the validity of x) V can *compute* after the interaction, V could have computed *without it*

Defining Zero Knowledge via Simulators

V's "view" = x + transcript τ + coins

- Formalised via "simulation paradigm": $View_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))$ can be *efficiently* simulated given only the instance



Definition 2 (Honest-Verifier *Perfect* ZK)

An IP Π is *honest-verifier perfect* ZK if there exists a PPT *simulator* Sim such that for *all* distinguishers D and all $x \in \mathcal{L}$, the following is zero

$$\Pr[\mathsf{D}(View_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(x)) = 1]$$

Exercise 2

What happens when one invokes the simulator on $x \notin \mathcal{L}$?

Plan for Today's Lecture...

What really constitutes a proof?

Interactive Proof (IP)

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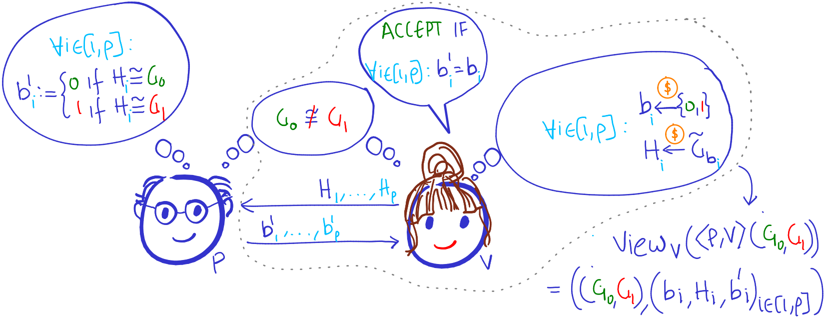
Zero-Knowledge IP

© Oded Goldreich

Main tools: simulation paradigm, Chernoff bound...

Theorem 2

Π_{GNI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GNI}$



Parallel/sequentially repeat to boost soundness

Π_{GNI} is *Honest-Verifier* ZK

Theorem 2

Π_{GNI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GNI}$

Proof.

$$\text{View}_V(\langle P,V\rangle(G_0,G_1)) := \left((G_0,G_1),(b_i,H_i,b'_i)_{i\in[1,\rho]}\right)$$

$\forall\, G_0 \not\cong G_1$:

$$\text{Sim}(G_0,G_1) := \forall i\in[1,\rho]: \text{sample } b_i \leftarrow \{0,1\} \text{ and } H_i \leftarrow \tilde{G}_{b_i}$$

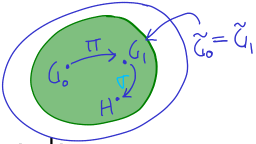
$$\text{o/p } \left((G_0,G_1),(b_i,H_i,b_i)_{i\in[1,\rho]}\right)$$

$\forall\, G_0 \not\cong G_1$: $\text{View}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\text{Sim}(G_0,G_1)$. □

Exercise 3

1. What happens if V is "malicious" and can deviate from protocol?
2. Using ideas from Π_{GNI}, build honest-verifier ZKP for $\mathcal{L}_{QNR}$

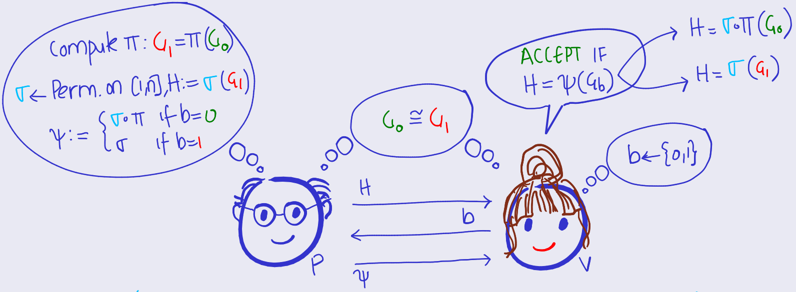
Honest-Verifier ZKP for GI...



Idea for ZK:

1. $G_0 \overset{\pi}{\cong} G_1 \Rightarrow$ if $G_1 \overset{\sigma}{\cong} H$ then $G_0 \overset{\sigma\circ\pi}{\cong} H$
2. Prover sends a random H s.t. $G_1 \overset{\sigma}{\cong} H$
3. Verifier asks to prove $G_0 \overset{\sigma\circ\pi}{\cong} H$ or $G_1 \overset{\sigma}{\cong} H$ at random

Protocol 2 (Π_{GI}: IP for GI)



(Parallel/sequentially repeat to boost soundness)

Theorem 3

Π_{GI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GI}$

Proof (idea for ZK: out of order sampling).

- Completeness: $G_0 \cong G_1 \Rightarrow$ P can answer both challenges $\Rightarrow$ V always accepts
- Soundness: $G_0 \not\cong G_1 \Rightarrow$ for *any* H P* commits to, $G_0 \cong H$ *and* $G_1 \cong H$ *cannot both hold* $\Rightarrow$ best P* can do is guess b
- Zero knowledge:

$$\mathrm{View}_V(\langle P,V\rangle(G_0,G_1)) := \left((G_0,G_1),\left(\overset{\sigma(G_1)}{H},\overset{\text{bit}}{b},\overset{\begin{cases}\sigma\circ\pi & \text{if } b=0\\ \sigma & \text{if } b=1\end{cases}}{\psi}\right)\right)$$

$\forall\, G_0 \overset{\pi}{\cong} G_1$: $\mathrm{Sim}(G_0,G_1)$: sample $b \leftarrow \{0,1\}$, $\psi \leftarrow$ permutation on $[1,n]$

set $H := \psi(G_b)$

o/p $((G_0,G_1),(H,b,\psi))$

$\forall\, G_0 \cong G_1$: $\mathrm{View}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\mathrm{Sim}(G_0,G_1)$. $\square$

Honest-Verifier ZKP for GI...

Exercise 4

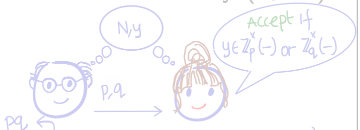
1. What happens if V is "malicious" and can deviate from protocol?
2. Using ideas from Π_{GI}, build honest-verifier ZKP for $\mathcal{L}_{QR}$

Which Languages have ZKPs? PSPACE Languages

Quadratic residuosity (QR)

$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^* \text{ s.t. } y = x^2 \bmod N\}$

Quad. non-residuosity (QNR)



$\mathcal{L}_{QNR} = \{(N,y) : \nexists x \in \mathbb{Z}_N^* \text{ s.t. } y = x^2 \bmod N\}$

Graph isomorphism (GI)

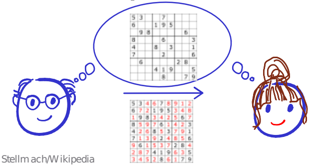
$\mathcal{L}_{GI} = \{(G_0, G_1) : \exists \text{permutation } \pi \text{ s.t. } G_1 = \pi(G_0)\}$

Graph non-isomorphism (GNI)

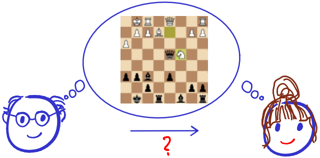
$\mathcal{L}_{GNI} = \{(G_0, G_1) : \nexists \text{permutation } \pi \text{ s.t. } G_1 = \pi(G_0)\}$

L19

Sudoku



Chess



Are Randomness and Interaction Necessary?

⇆ Interaction is necessary

Fact 4

If $\mathcal{L}$ has a non-interactive (i.e, one-message) ZKP then $\mathcal{L}$ is "trivial"

Randomness is necessary

Exercise 5

If $\mathcal{L}$ has an IP with deterministic verifier then $\mathcal{L} \in \mathrm{NP}$

Fact 5

If $\mathcal{L}$ has an ZKP with deterministic verifier then $\mathcal{L}$ is "trivial"

Recap/Next Lecture

- Traditional "NP" proofs vs *interactive* proofs
  - IP is more powerful: IP for GNI
- Zero-knowledge proofs
  - Knowledge vs. information
  - Modelled "zero knowledge" via simulation paradigm
- (Honest-verifier) ZKP for GNI (A5: QNR) and GI (A5: QR)

- Next Lecture:
  - Computational ZKP for all of NP!
    - New cryptographic primitive: commitment schemes
  - Return to ID protocols: zero-knowledge proof of knowledge

References

1. [Gol01, Chapter 4] for details of today's lecture
2. [GMR89] for definitional and philosophical discussion on ZK
3. The ZKPs for GI and GNI are taken from [GMR89, GMW91]
4. IP for all of PSPACE is due to [LFKN92, Sha90]. Computational ZKP for all of PSPACE is due to [GMW91].

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