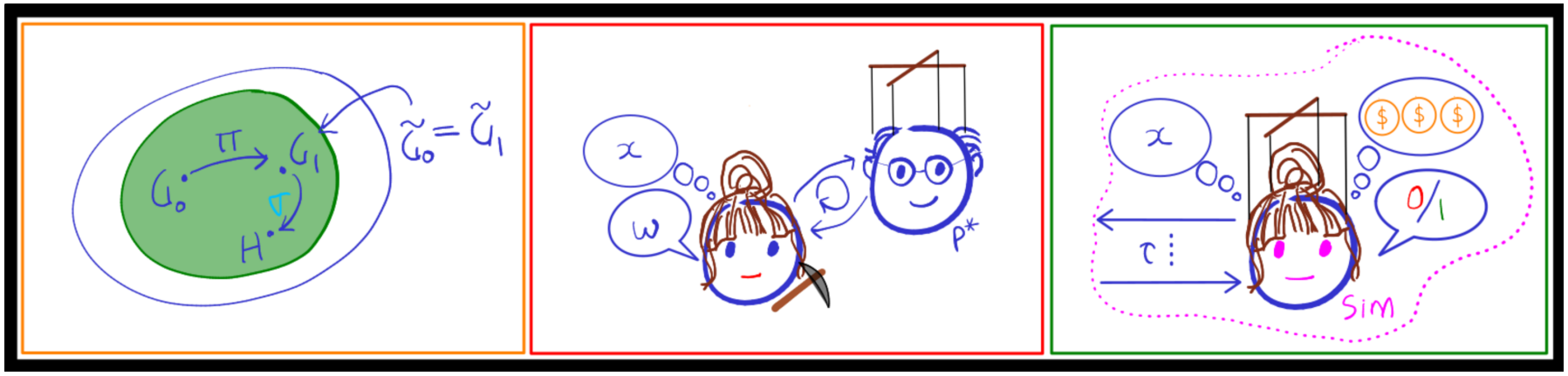
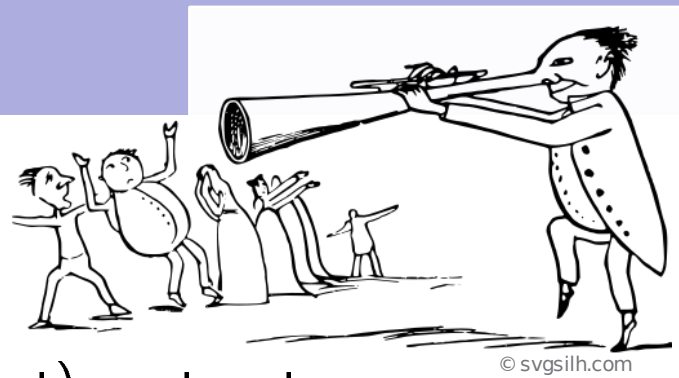


CS409m: Introduction to Cryptography

Lecture 19 (22/Oct/25)

Instructor: Chethan Kamath

Announcements



- Feedback form for course (post mid-sem part) sent out
- Assignment 5 out yesterday (21/Oct)

- Quiz 2 viewing on 24/Oct (Friday), 12:30-14:30
  - Submit your cribs online by 29/Oct (next Wednesday)
- Quiz 3 on 29/Oct (next Wednesday)
  - 08:25-09:25, in CC103/CC105

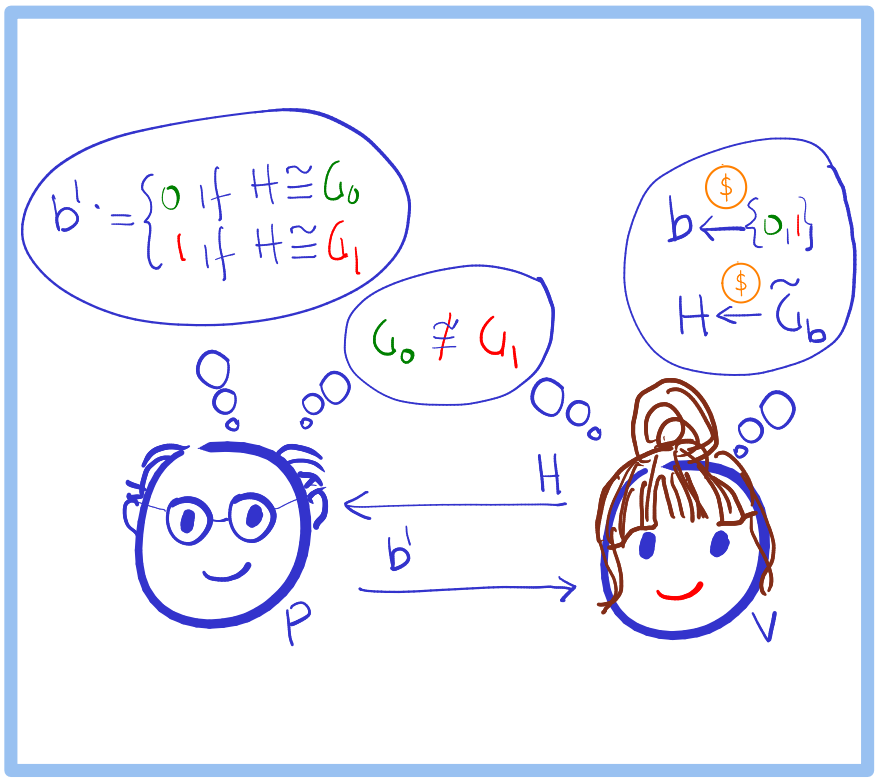
- Lab Exercise 4 will be released today (22/Oct)
  - Submit flag by 29/Oct EoD (Wednesday)
  - Submit write-up by 31/Oct EoD (Friday)

Recall from Last Lecture...

- Interactive proofs vs NP proofs:
  - Prover convinces verifier using *interaction*
  - Verifier is *random*

Interactive Proof (IP)

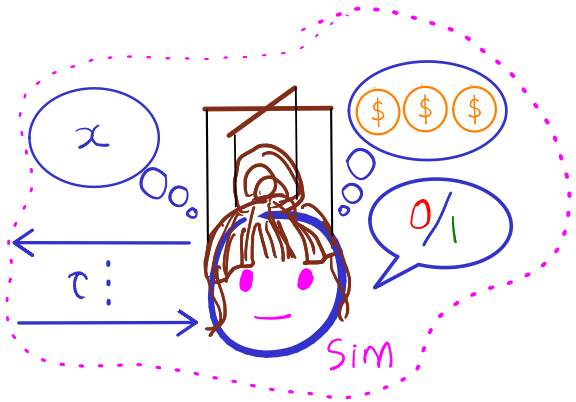
IP for GNI



Plan for Today's Lecture...

IP where prover reveals *no non-trivial knowledge* to the verifier

NEW Zero-knowledge (ZK) IP



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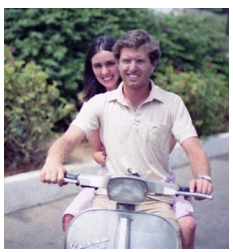

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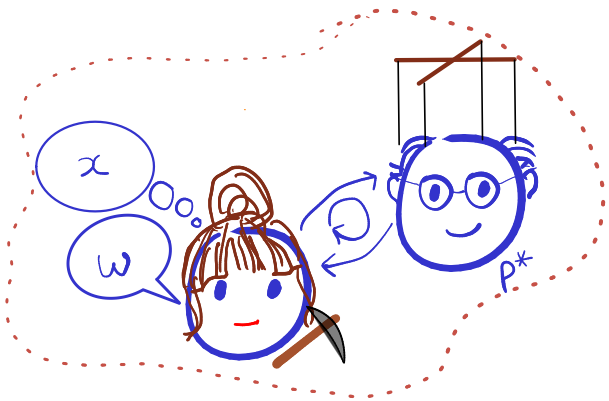
ZK Proof of Knowledge (PoK) NEW



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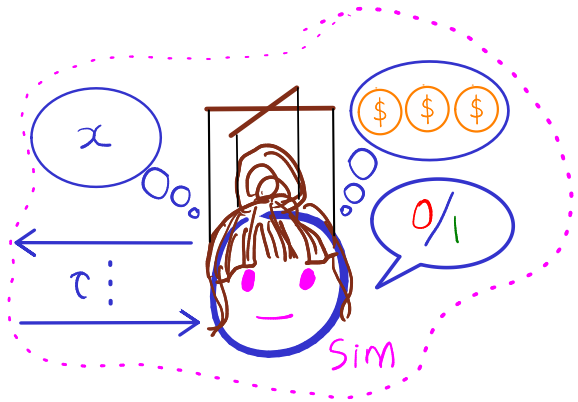

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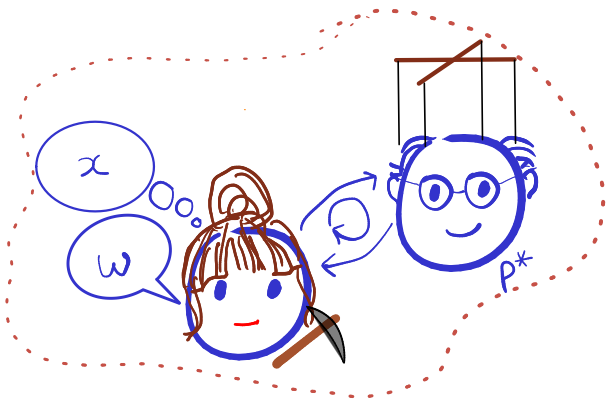

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Main tools: simulator and extractor

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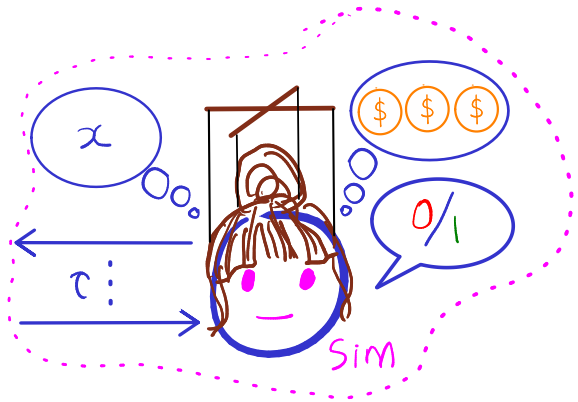

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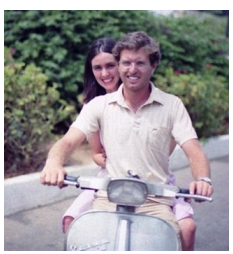

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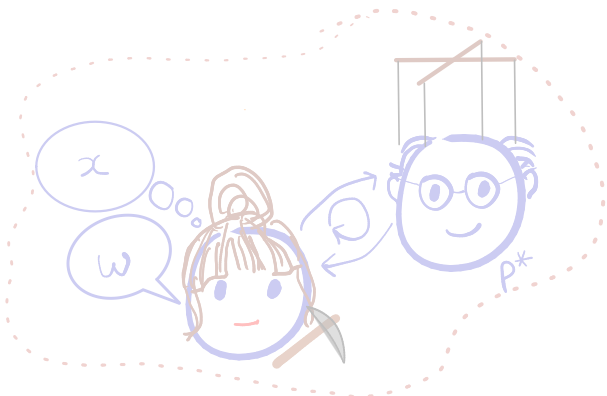

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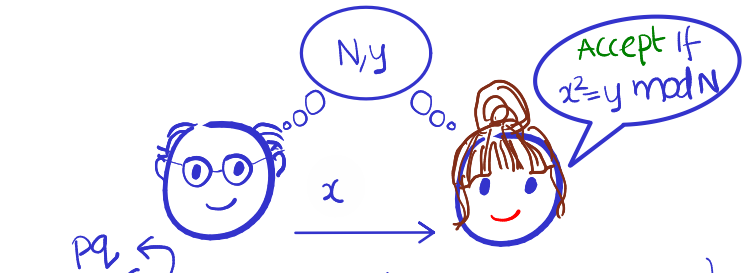
Main tools: simulator and extractor

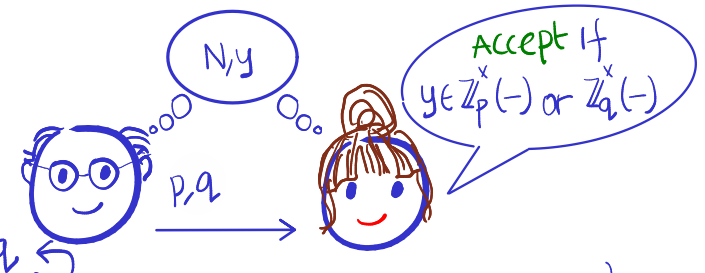
The NP Proofs We Saw Leaked Information

Quadratic residuosity (QR)



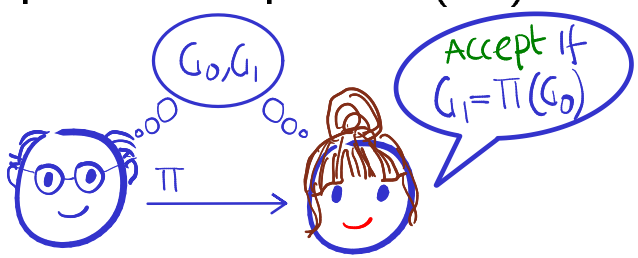
$$\alpha_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$$

Quad. non-residuosity (QNR)



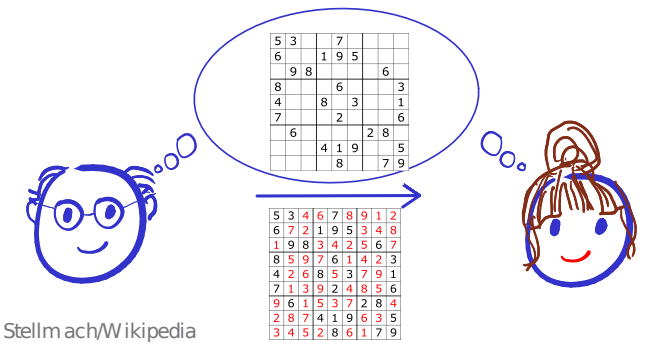
$$\alpha_{QNR} = \{(N,y) : \nexists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$$

Graph isomorphism (GI)



$$\alpha_{GI} = \{(G_0, G_1) : \exists \text{permutation } \pi \text{ s.t. } G_1 = \pi(G_0)\}$$

Sudoku



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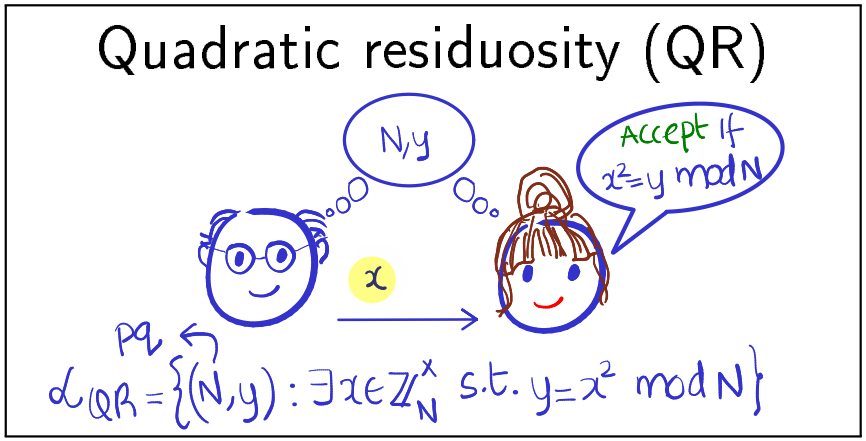
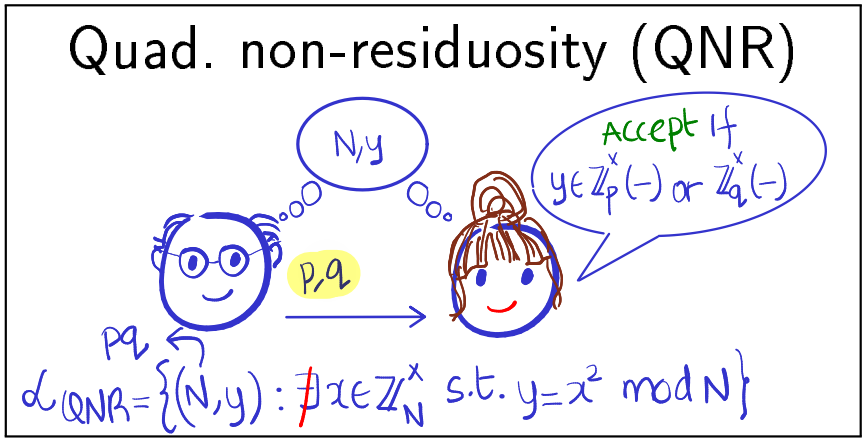
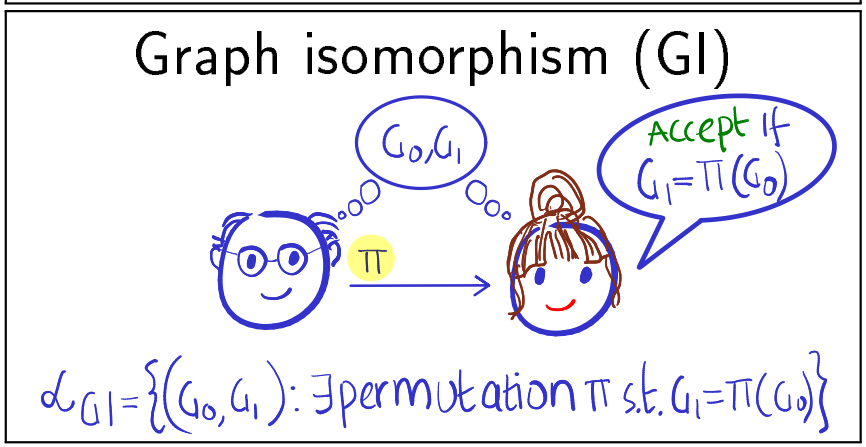
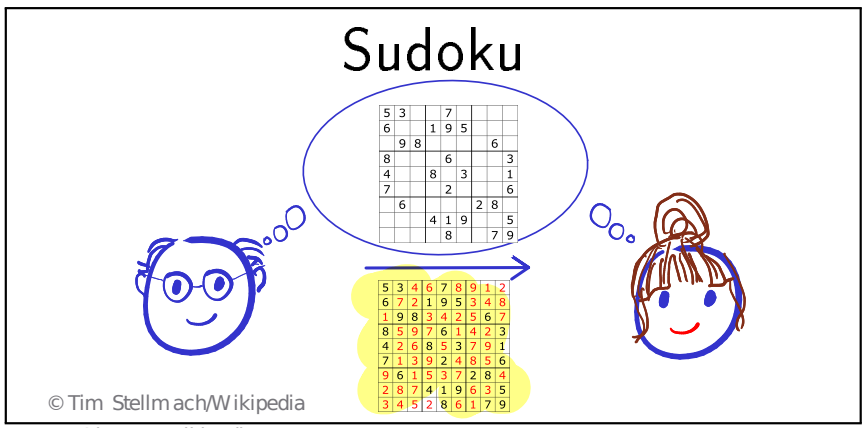
The NP Proofs We Saw Leaked Information









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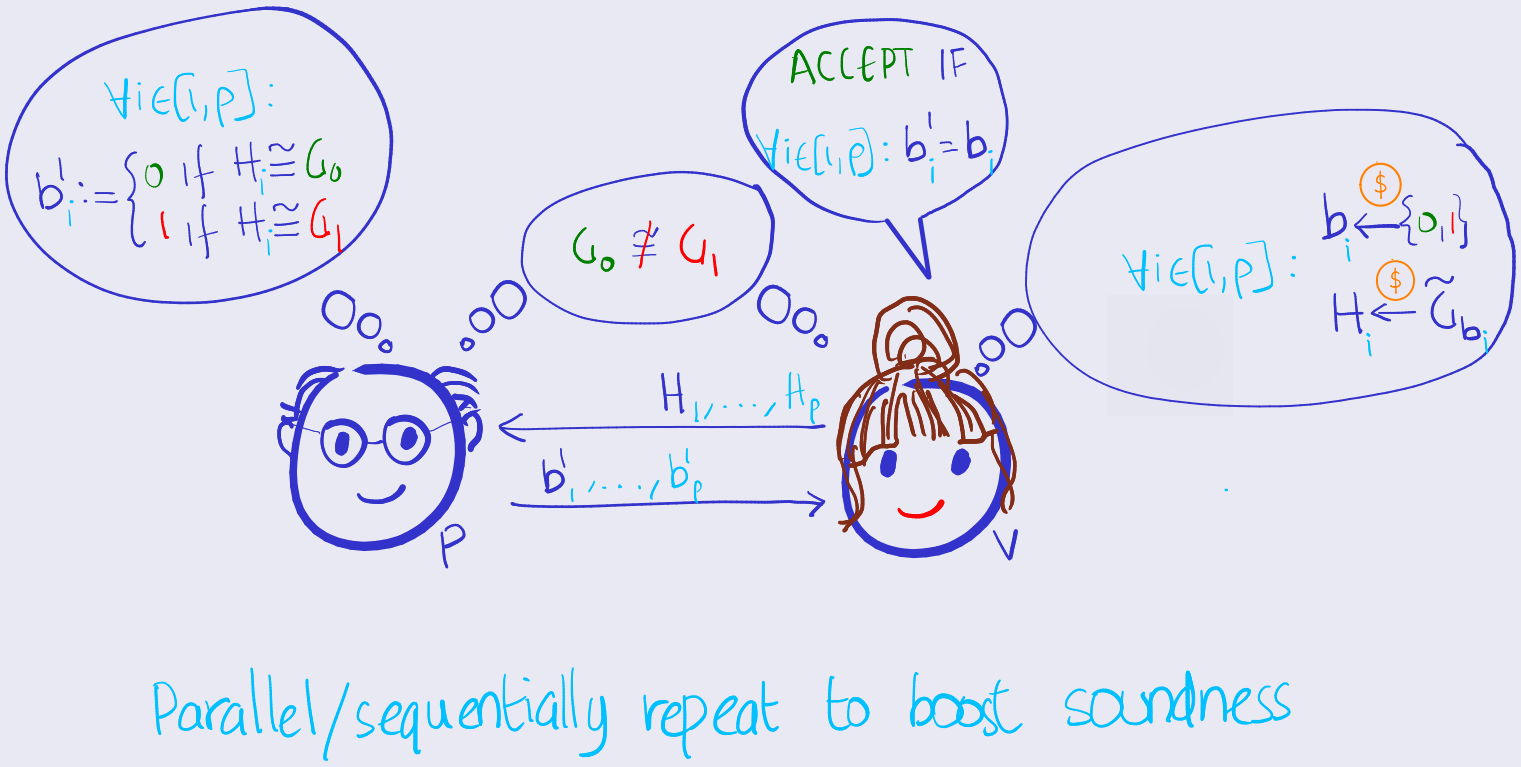
- Verifier gains "non-trivial knowledge" about witness w
  - Not desirable, e.g., when $x = pk$ and $w = sk$ (identification)

But the IP for GNI We Saw Doesn't Seem to

Protocol 1 (Π_{GNI}: IP for GNI)



Parallel/sequentially repeat to boost soundness

- Seems V *gains no knowledge* beyond validity of the statement
- We will show that Π_{GNI} is (honest-verifier) zero-knowledge!

How to Capture "V Gains No Knowledge"?

Knowledge vs. information ← in the information-theoretic sense

- Knowledge is computational

How to Capture "V Gains No Knowledge"?

Knowledge vs. information

- Knowledge is computational: e.g., consider NP proof for GI
  - Given (G_0, G_1), the isomorphism π contains no *information*
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- Knowledge pertains to public objects:
  - Flipping a *private* fair coin b and (later) revealing its outcome leads to V gaining *information*
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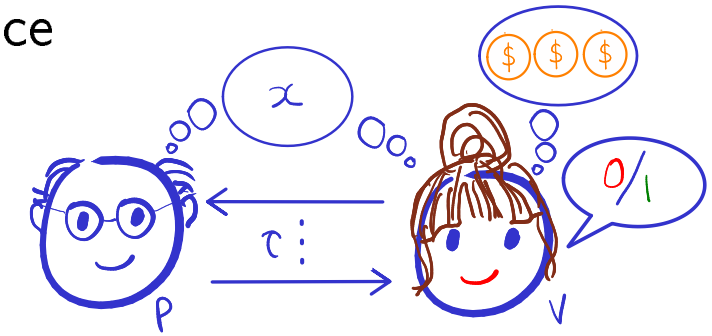
Intuitively, "V gains no knowledge" if anything V can *compute* after the interaction, V could have computed *without it*

Defining Zero Knowledge via Simulators...

- Formalised via "simulation paradigm": $\mathit{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))$ can be *efficiently* simulated given only the instance



Defining Zero Knowledge via Simulators...

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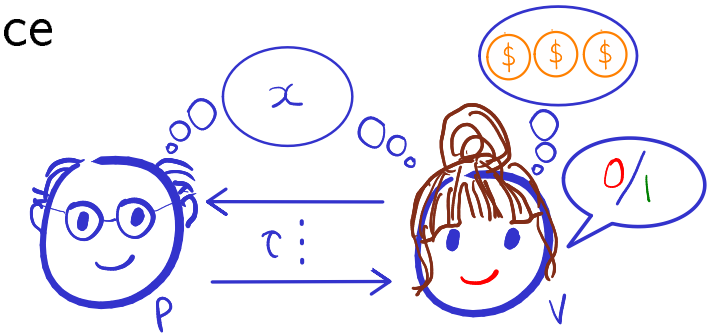
V's "view" = x + transcript τ + coins



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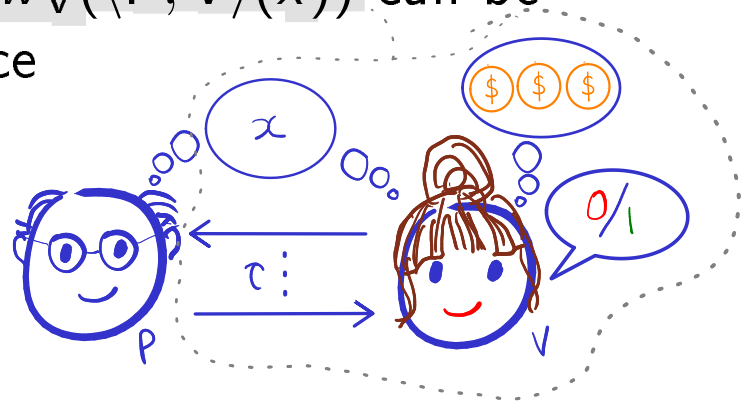

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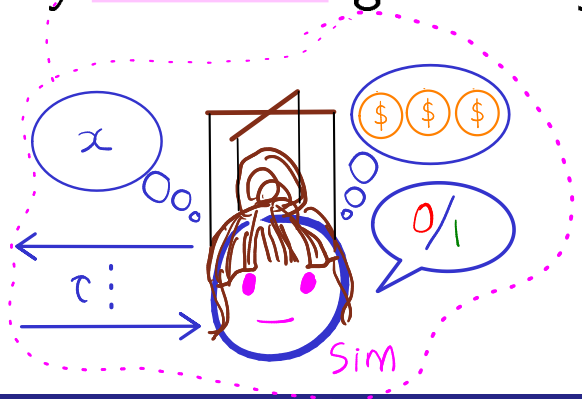

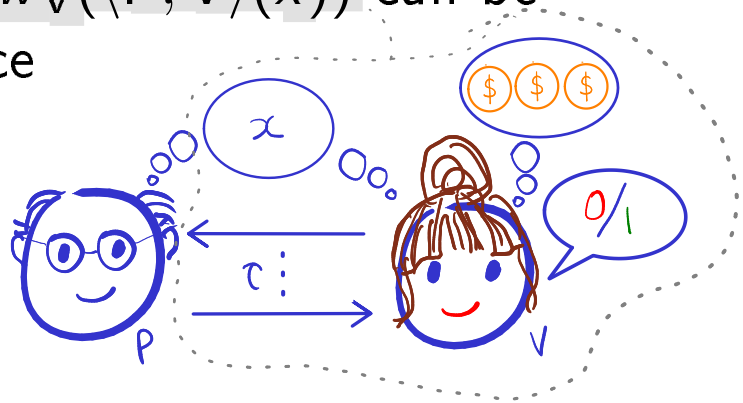

Definition 1 (Honest-Verifier *Perfect* ZK)

An IP Π is *honest-verifier perfect* ZK if there exists a PPT *simulator* Sim such that for *all* distinguishers D and all $x \in \mathcal{L}$

$$\Pr[\mathsf{D}(\mathit{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))) = 1] = \Pr[\mathsf{D}(\mathsf{Sim}(x)) = 1]$$

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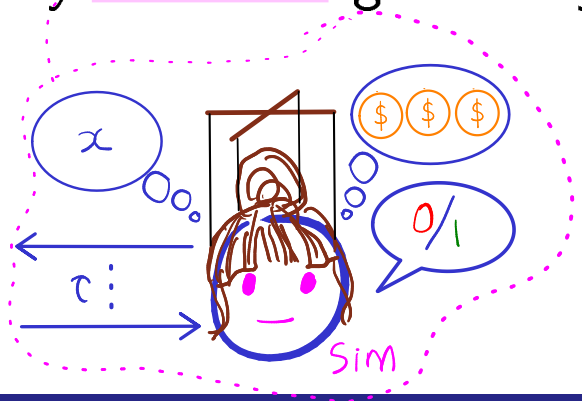

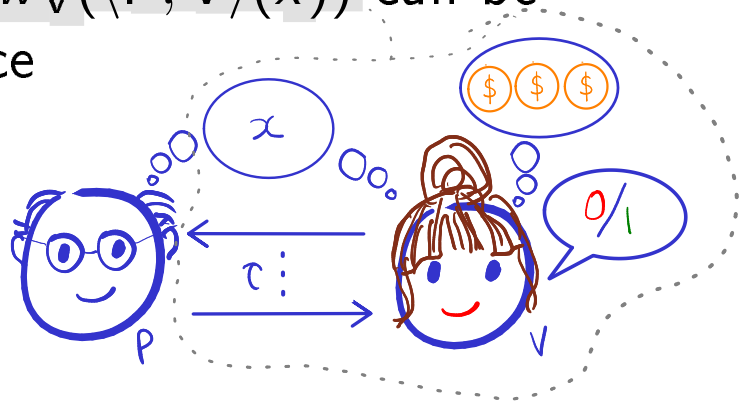

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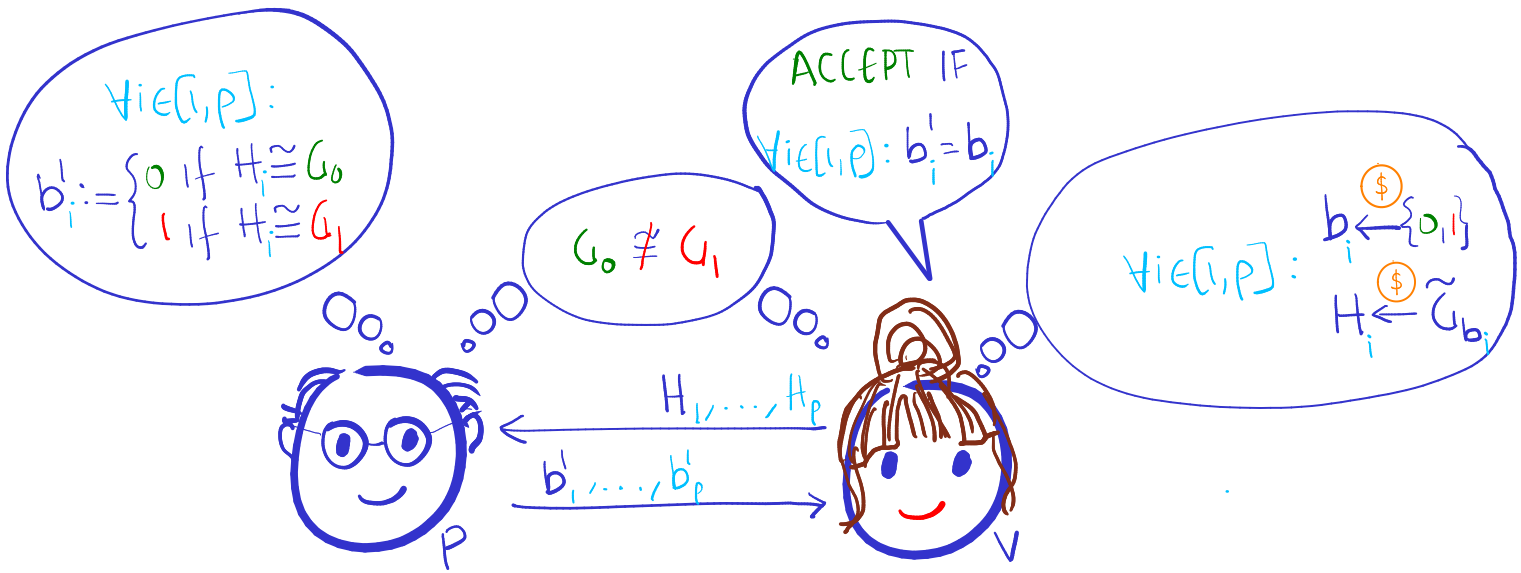
Exercise 1

What happens when one invokes the simulator on $x \notin \mathcal{L}$?

Π_{GNI} is *Honest-Verifier* ZK

Theorem 1

Π_{GNI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GNI}$



Parallel/sequentially repeat to boost soundness

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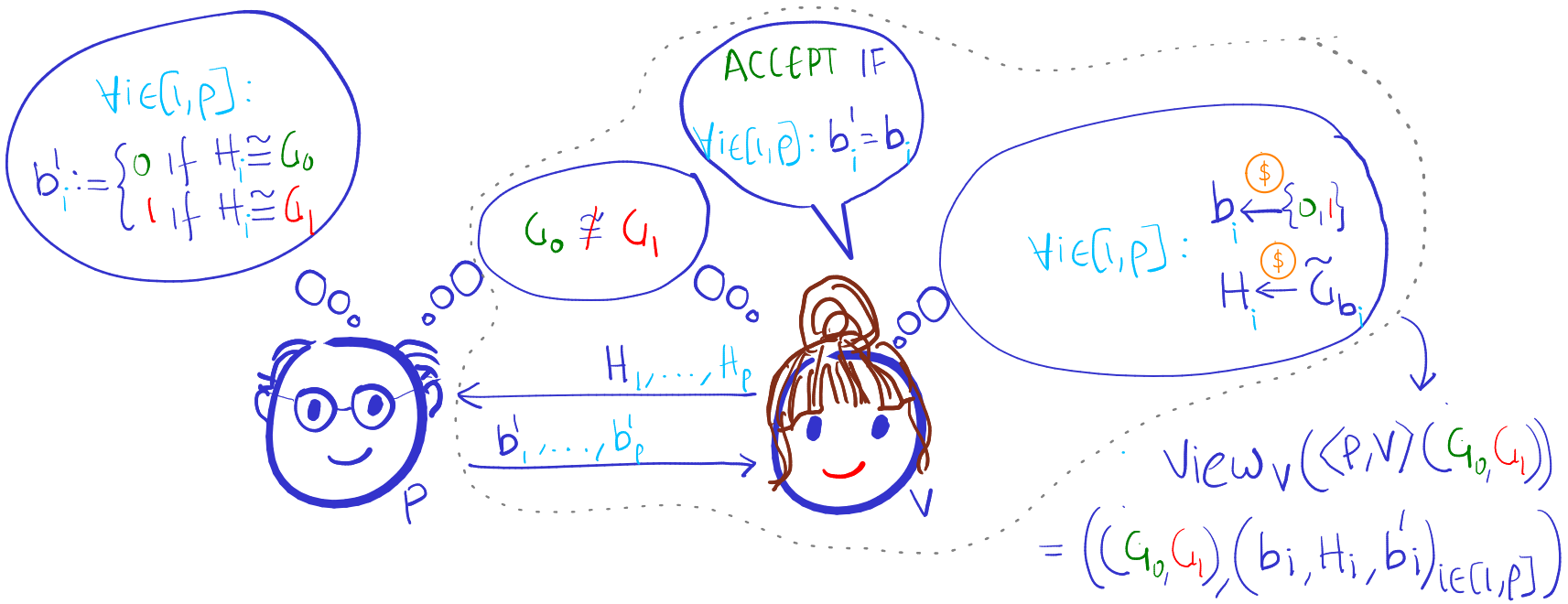

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Proof.

$$\text{view}_V(\langle P, V\rangle)(G_0, G_1) := \left((G_0, G_1), \left(b_i, H_i, b_i'\right)_{i\in[1,p]}\right)$$

$\forall\, G_0 \not\cong G_1$:

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$\forall\ G_0 \not\cong G_1$: $\text{View}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\text{Sim}(G_0,G_1)$.

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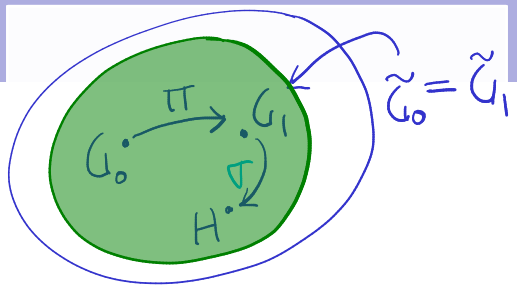
Exercise 2

1. What happens if V is "malicious" and can deviate from protocol?
2. Using ideas from Π_{GNI}, build honest-verifier ZKP for $\mathcal{L}_{QNR}$

Honest-Verifier ZKP for GI...

Idea for ZK: $G_0 \overset{\pi}{\cong} G_1 \Rightarrow$ if $G_1 \overset{\sigma}{\cong} H$ then $G_0 \overset{\sigma\circ\pi}{\cong} H$



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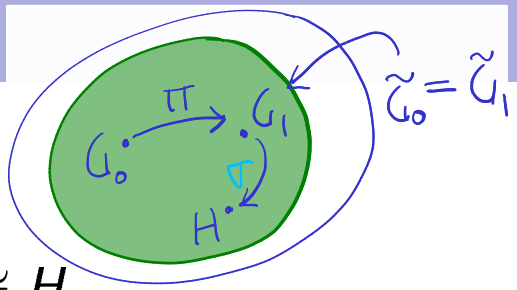

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1. Prover "commits" by sending random H s.t. $G_1 \cong H$
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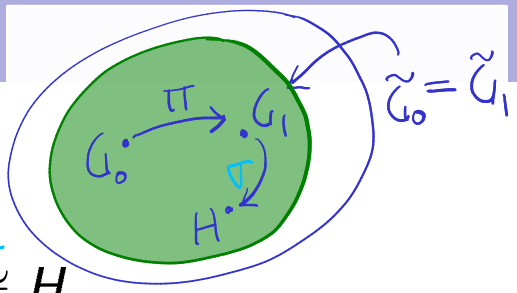

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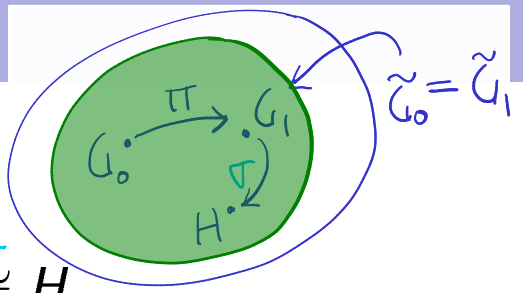

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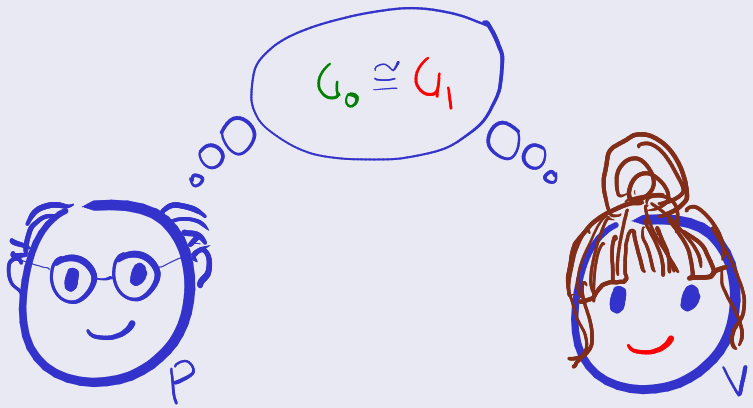
Protocol 2 (Π_{GI}: IP for GI)



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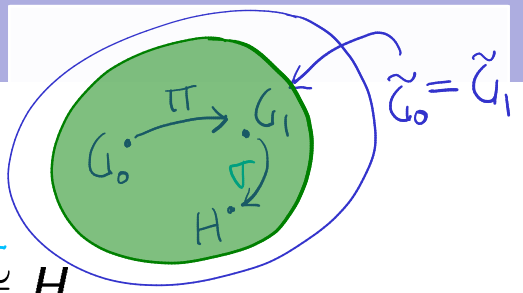

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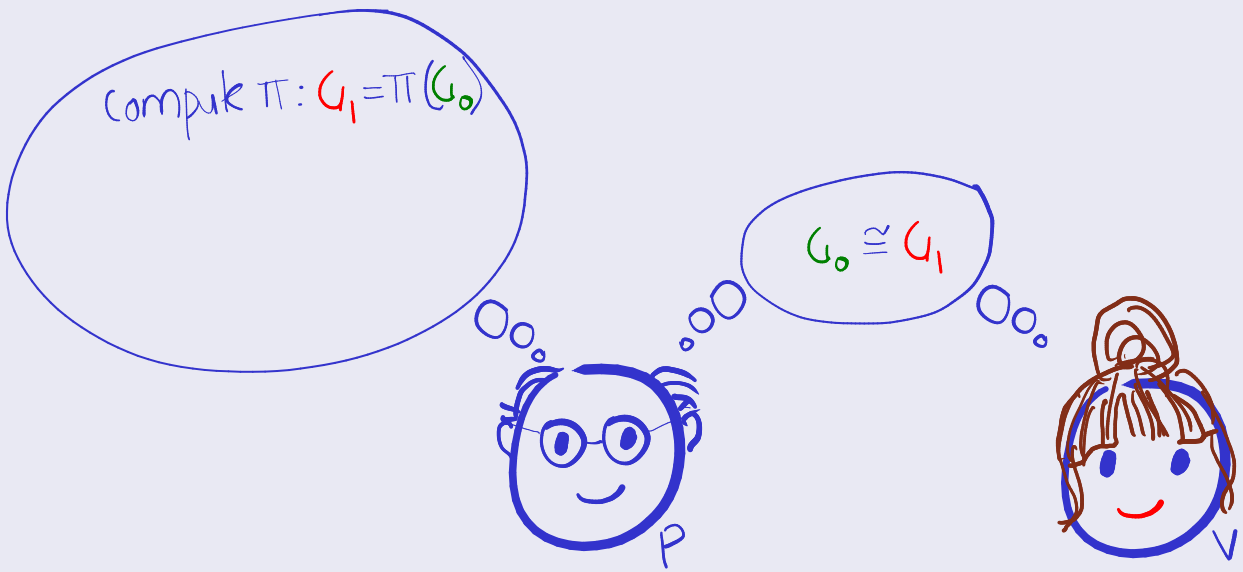
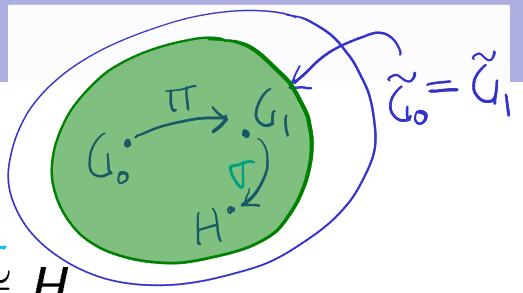


Honest-Verifier ZKP for GI...



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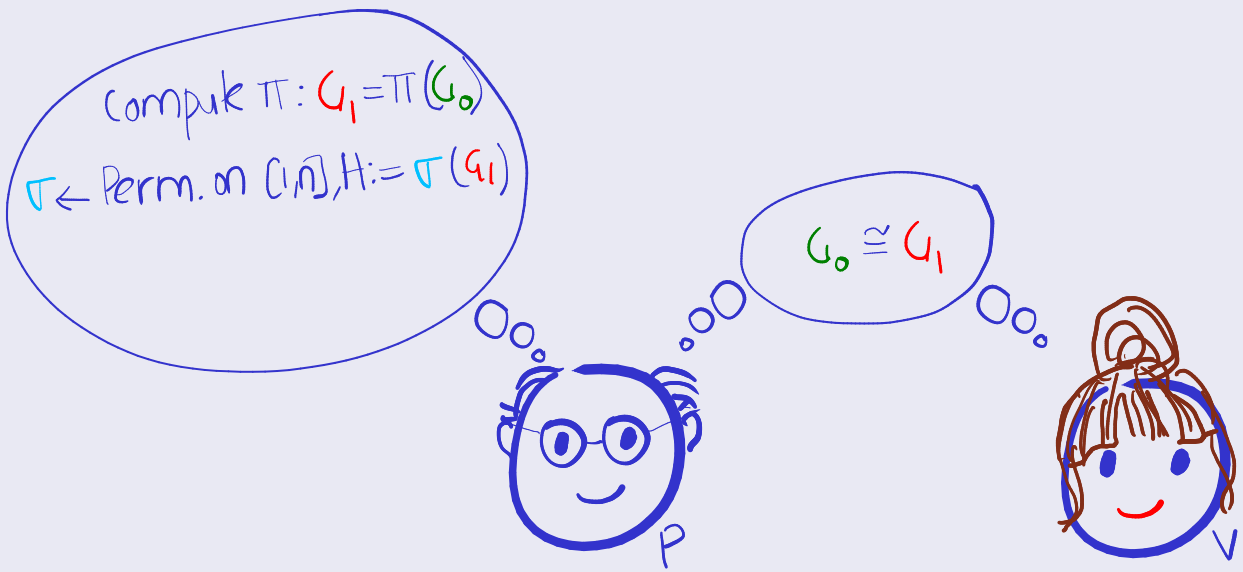

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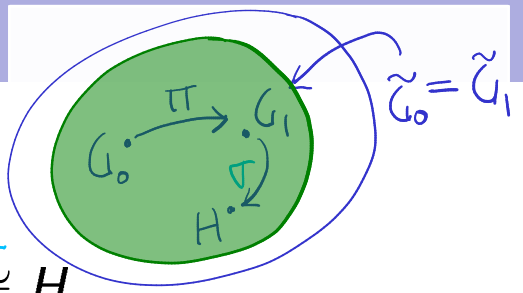

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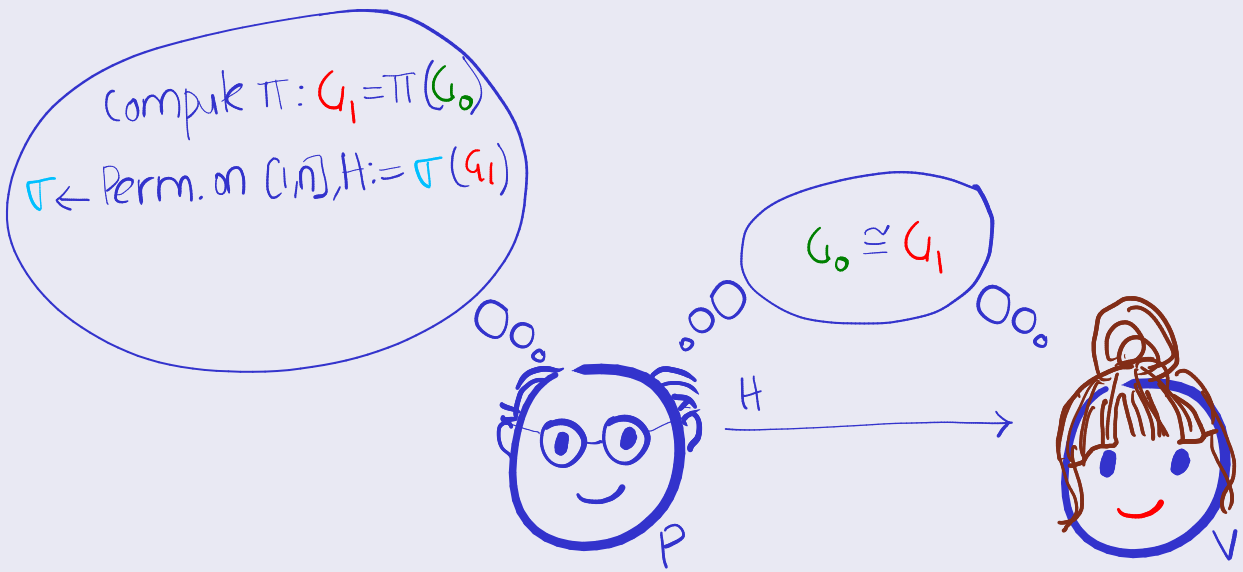

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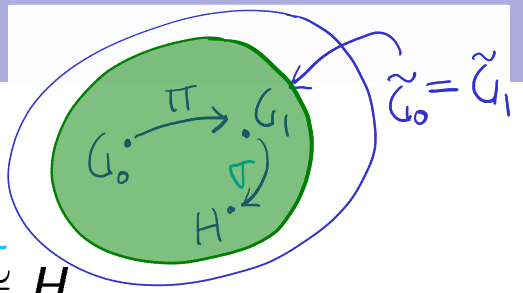

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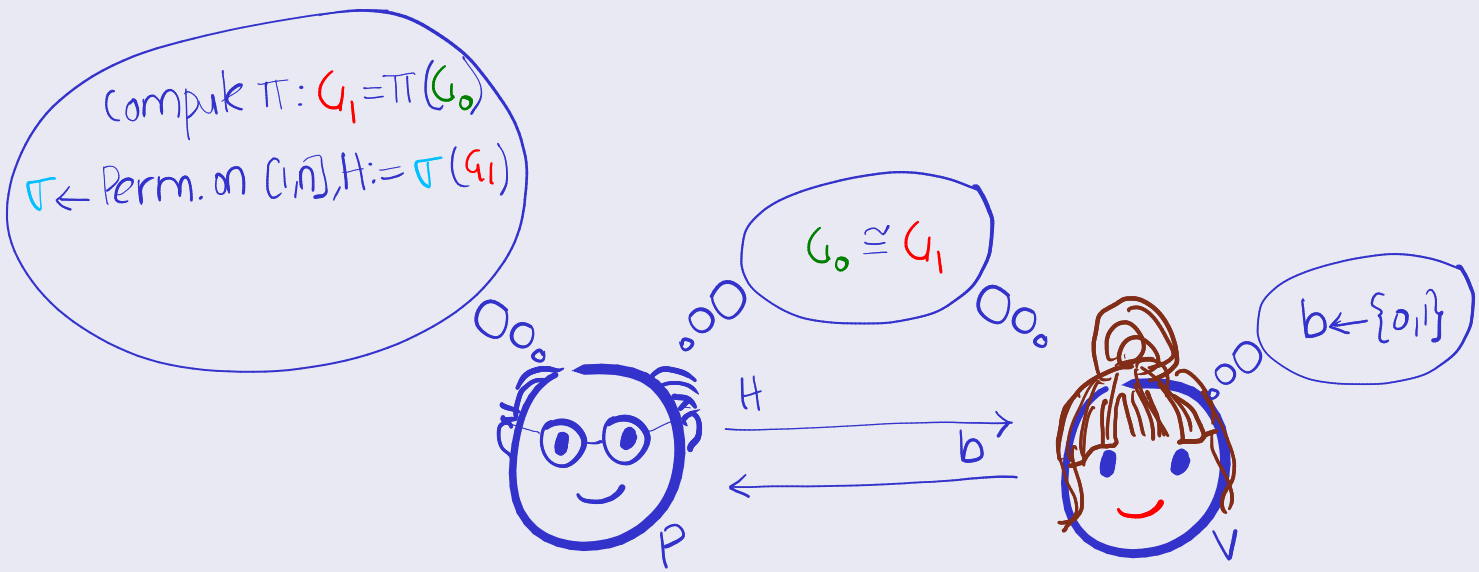

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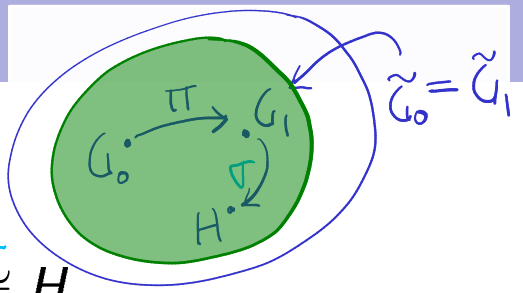

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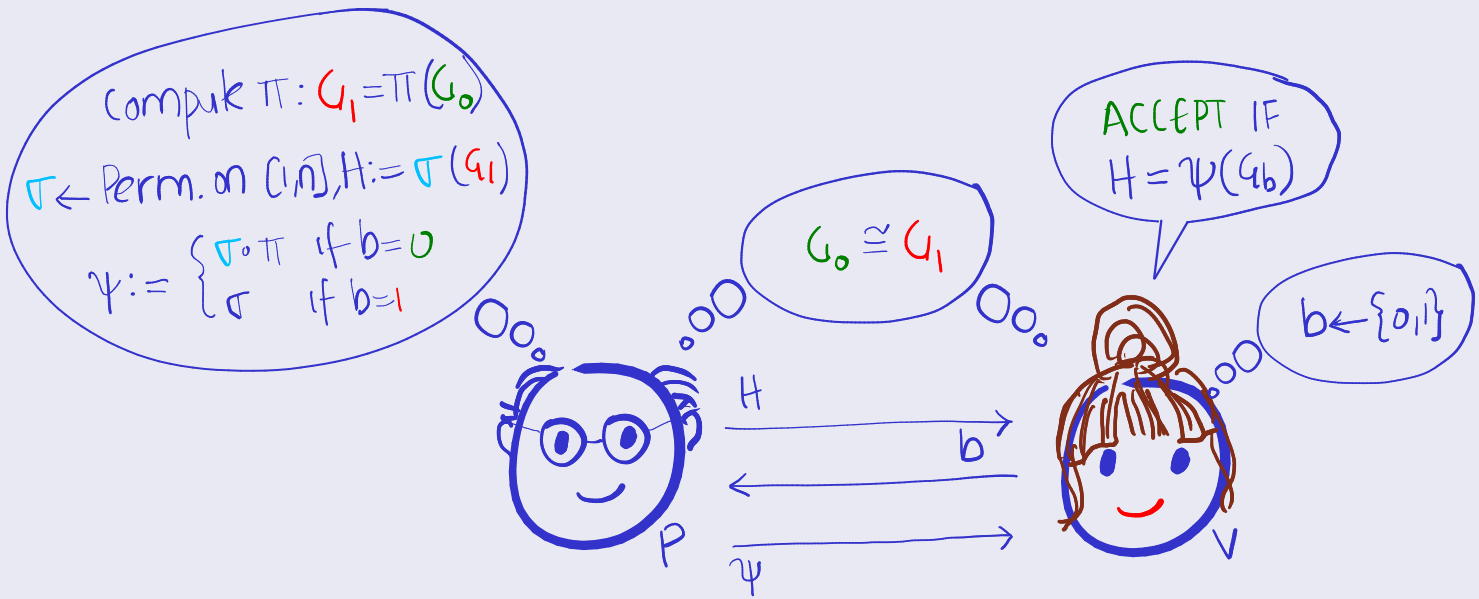

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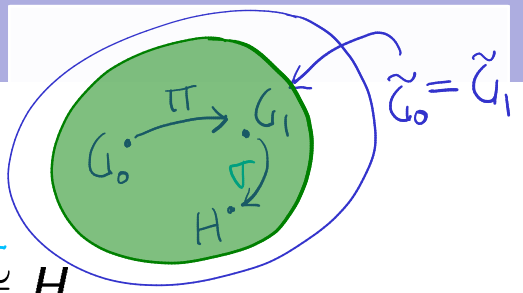

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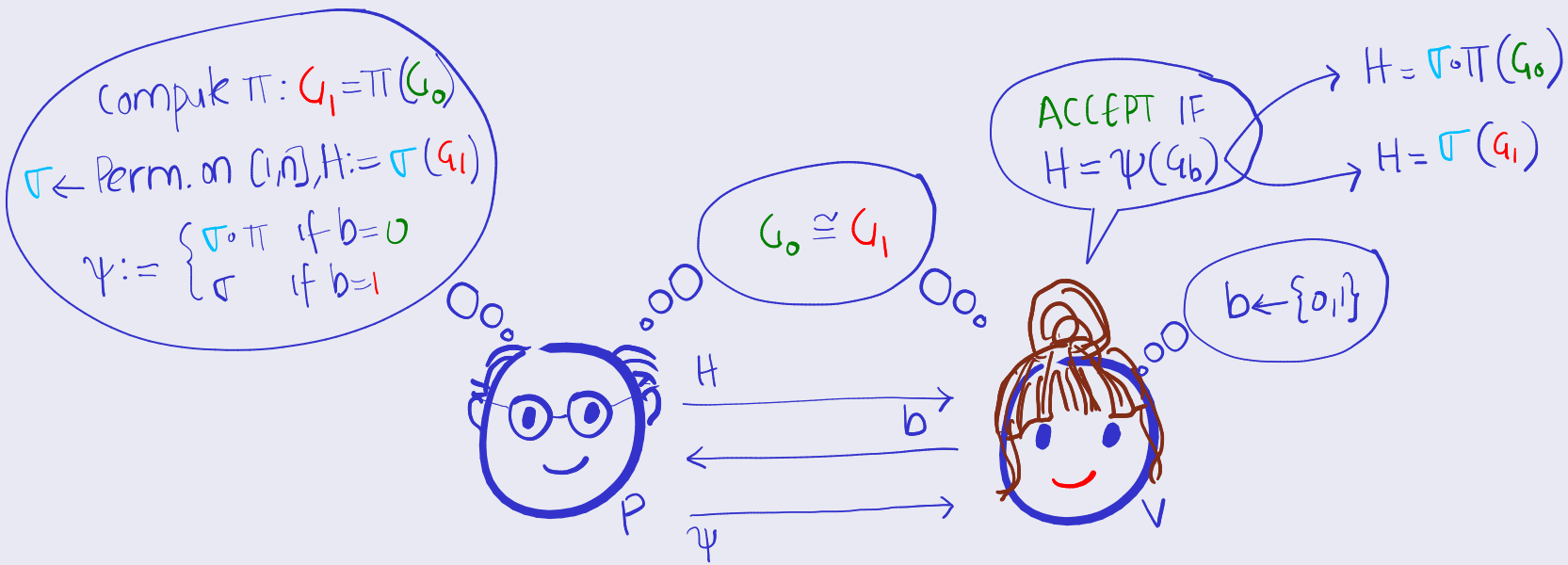

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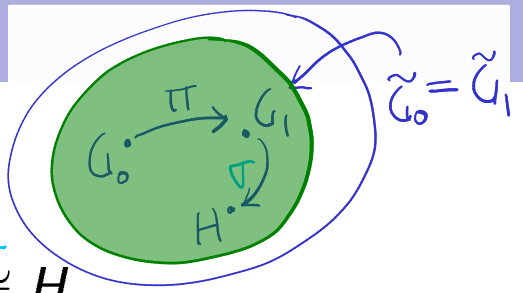

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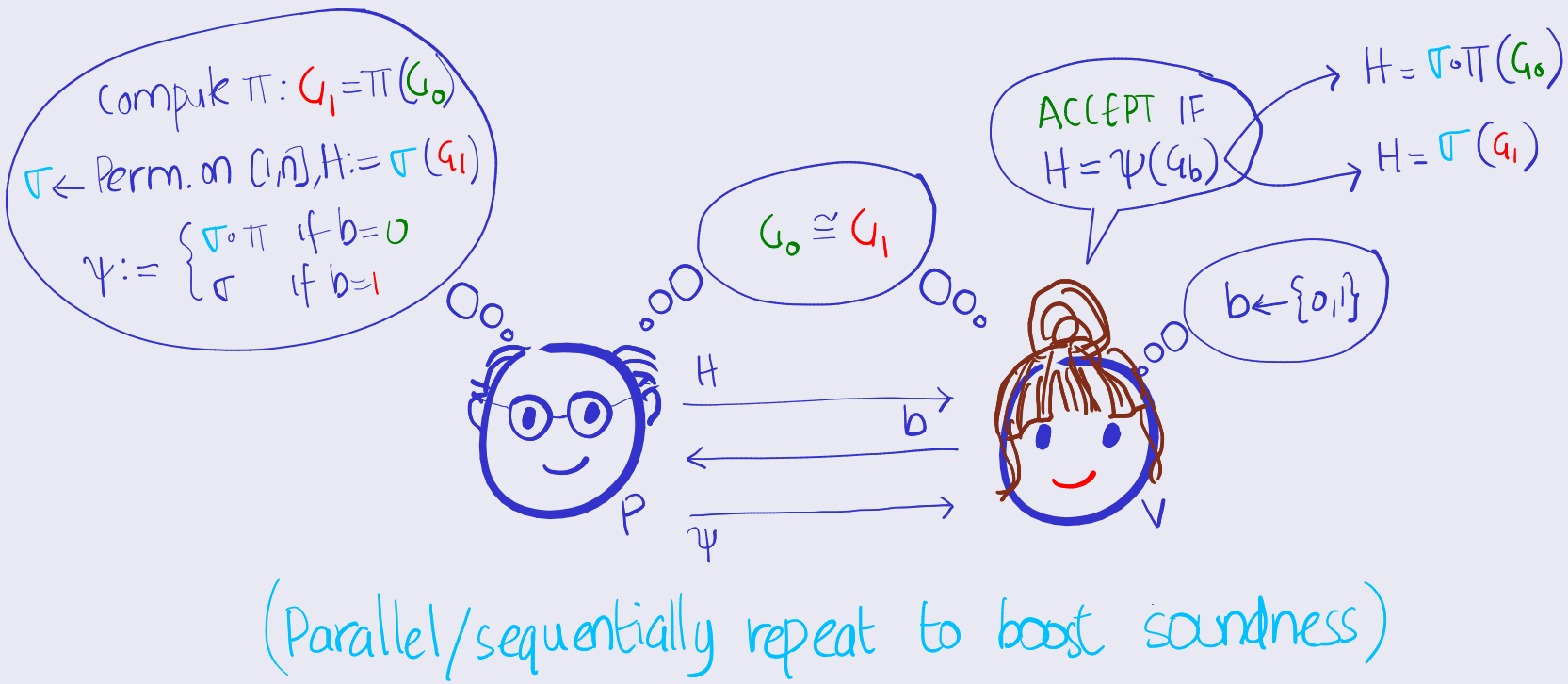

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- Completeness: $G_0 \cong G_1 \Rightarrow$ P can answer both challenges $\Rightarrow$ V always accepts
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$$\mathrm{View}_V(\langle P,V\rangle(G_0,G_1)) := ((G_0,G_1),(H,b,\psi))$$

$H = \sigma(G_1)$, b = bit, $\psi = \begin{cases} \sigma \circ \pi & \text{if } b=0 \\ \sigma & \text{if } b=1 \end{cases}$

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□

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$\forall\ G_0 \overset{\pi}{\cong} G_1$: $\text{sim}(G_0,G_1)$: sample $b \leftarrow \{0,1\}$, $\psi \leftarrow$ permutation on $[1,n]$

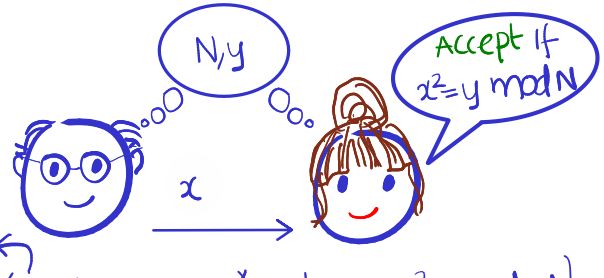
set $H := \psi(G_b)$

o/p $\left((G_0,G_1),\left(H,b,\psi\right)\right)$

$\forall\ G_0 \cong G_1$: $\text{view}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\text{sim}(G_0,G_1)$. □

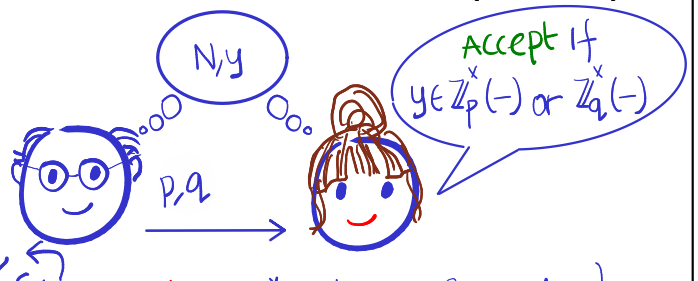
Which Languages have ZKPs?

Quadratic residuosity (QR)



$$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^{\times} \text{ s.t. } y = x^2 \bmod N\}$$

Quad. non-residuosity (QNR)



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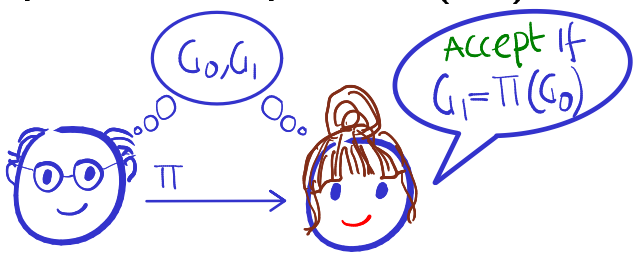

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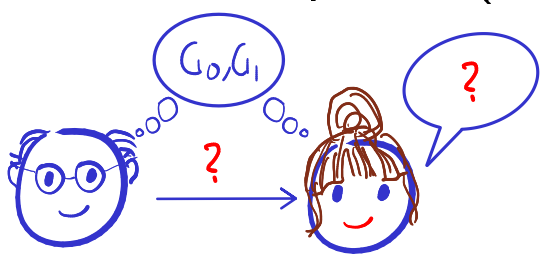

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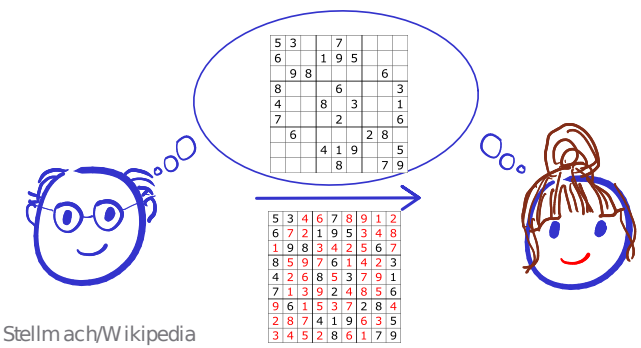
Sudoku



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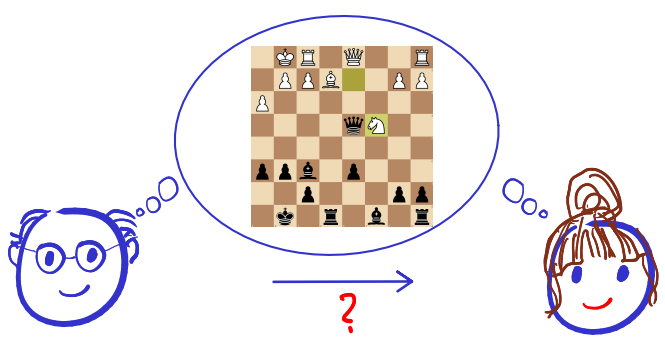
Chess



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Which Languages have ZKPs?

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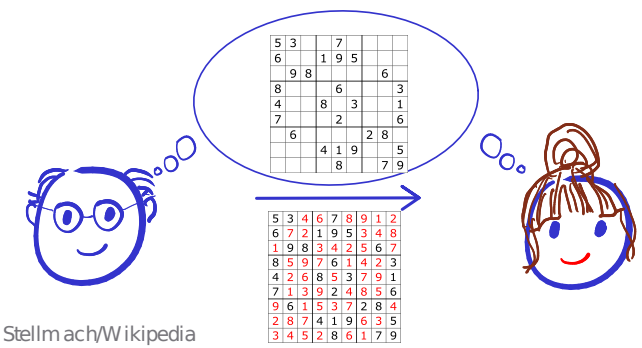

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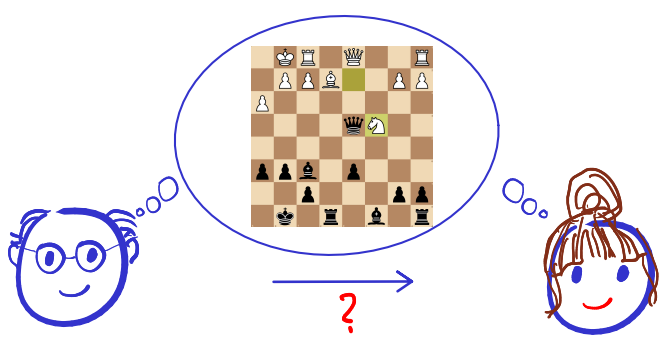

L19

Which Languages have ZKPs? PSPACE Languages

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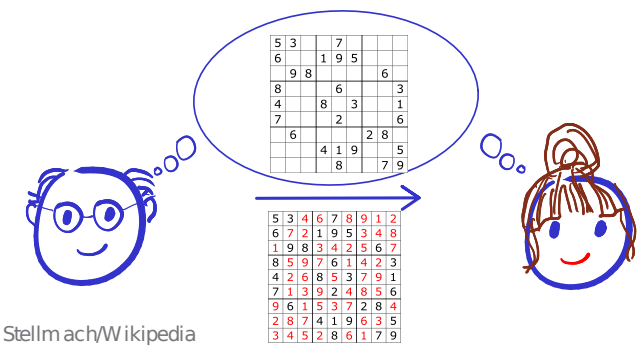

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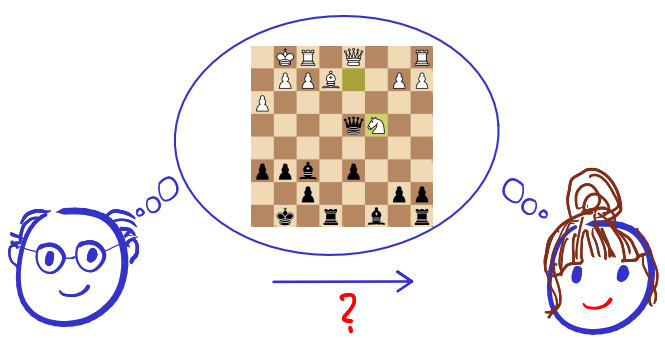

L19

Are Randomness and Interaction Necessary?

$\leftrightarrows$ Interaction is necessary

Fact 3

If $\mathcal{L}$ has a non-interactive (i.e, one-message) ZKP then $\mathcal{L}$ is "trivial"

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Randomness is necessary

Exercise 3

If $\mathcal{L}$ has an IP with deterministic verifier then $\mathcal{L} \in$ NP

Fact 4

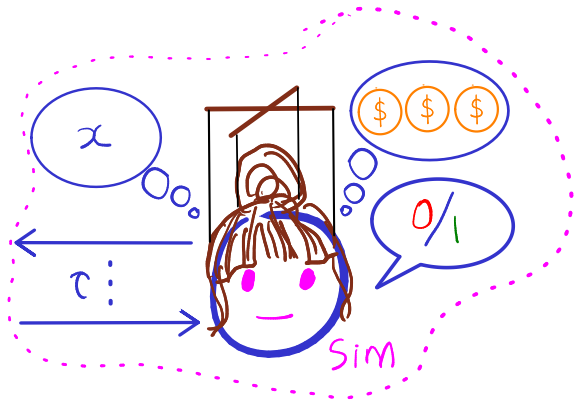
If $\mathcal{L}$ has an ZKP with deterministic verifier then $\mathcal{L}$ is "trivial"

Plan for Today's Lecture...

IP where prover reveals *no non-trivial knowledge* to the verifier

NEW Zero-knowledge (ZK) IP

ZK Proof of Knowledge (PoK) NEW



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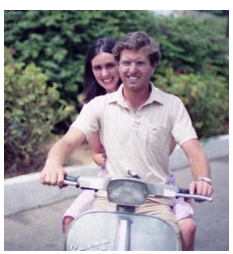

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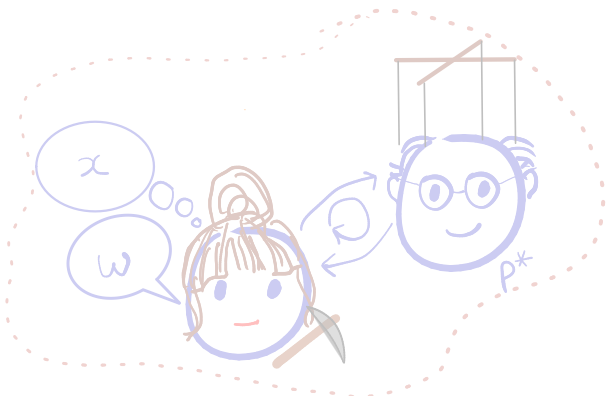

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Main tools: simulator and extractor

Sometimes Stronger Guarantees than ZK Needed

- Recall ZK IP requirements:
  1. Completeness
  2. Soundness
  3. Zero-knowledge (ZK)

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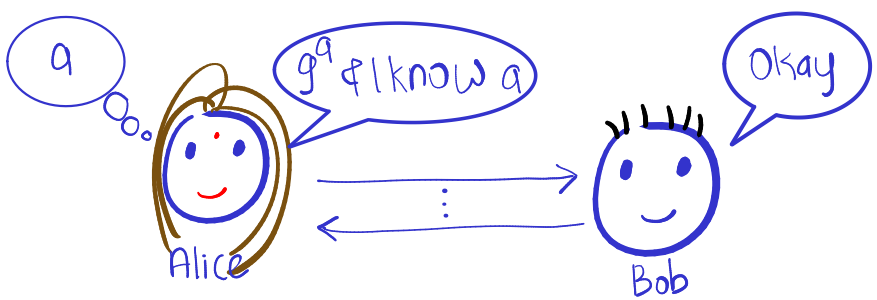
- Sometimes V needs to be convinced that P *knows* a witness
- E.g. Identification for ElGamal PKE in cyclic group $\mathbb{G}$
  - Public key is $h := g^a$ and secret key is the discrete log a
  - Owner has to prove they *possess* a (such an a always exists)



How to Quantify Knowledge?

- For defining ZK, we only quantified "gain of knowledge"
  - "V gains no knowledge" if anything V can *compute* after the interaction with P, it could have computed *without it*
  - Formalised via simulator: V's view can be *efficiently simulated* given only the instance x

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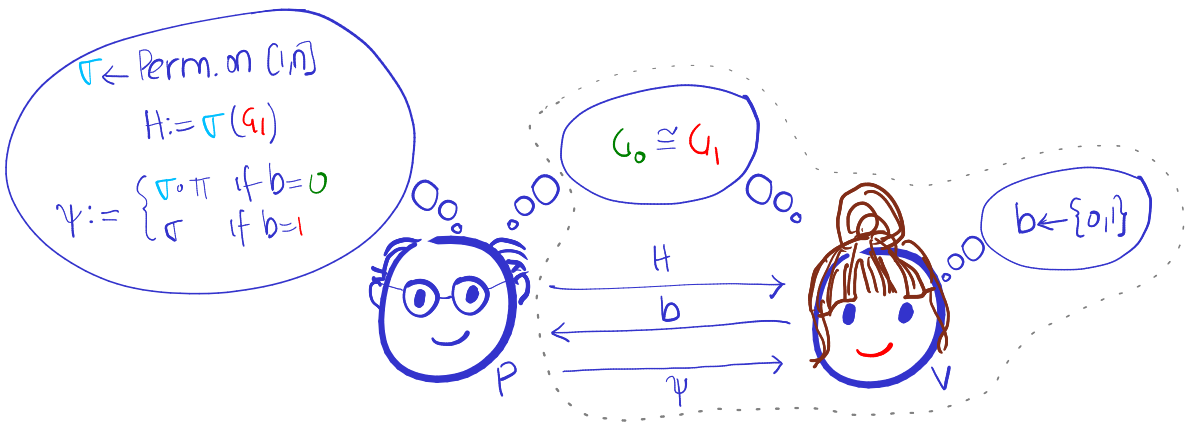

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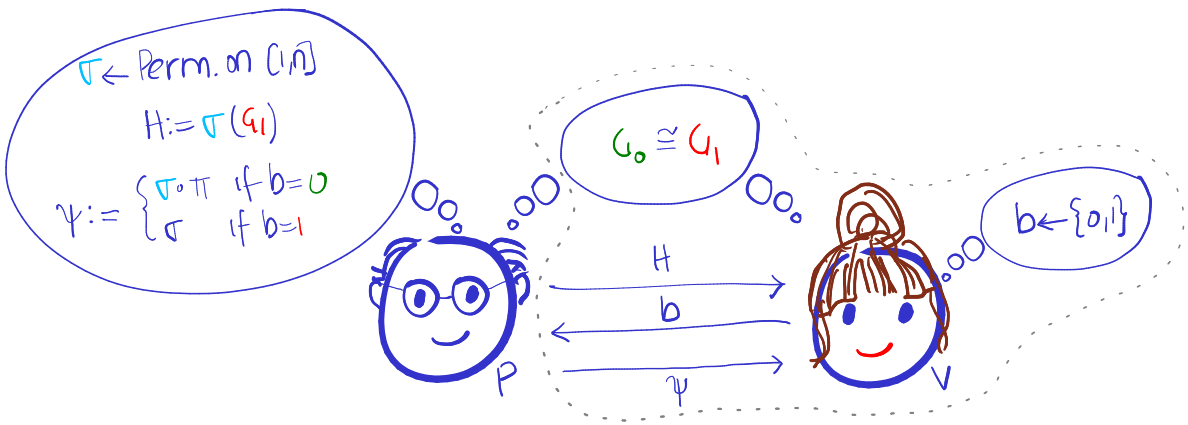


  - For P in Π_{GI}? Should be possible to *efficiently extract* isomorphism π given access to P
  - In general, for NP: should be possible to extract a witness w

Let's Define ZK *Proof of Knowledge*

Definition 2 (ZKPoK)

An interactive protocol $\Pi = (\mathsf{P}, \mathsf{V})$ for an NP language $\mathcal{L}$ is a zero-knowledge *proof of knowledge* if it is

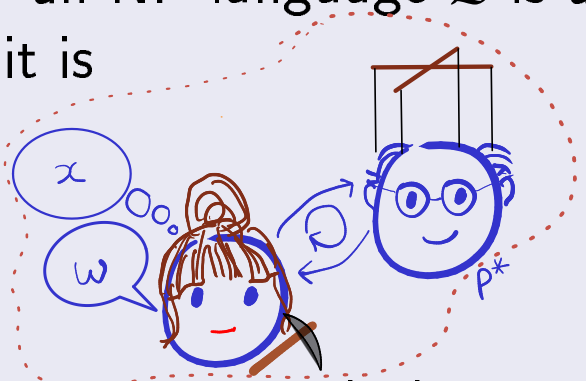
1. Complete
2. Zero knowledge

Let's Define ZK *Proof of Knowledge*

Definition 2 (ZKPoK)

An interactive protocol $\Pi = (\mathsf{P}, \mathsf{V})$ for an NP language $\mathcal{L}$ is a zero-knowledge *proof of knowledge* if it is

1. Complete
2. Zero knowledge
3. Knowledge sound:
   - $\exists$ expected polynomial-time *extractor* Ext such that
   - $\forall$ prover P^* and instance x:



$$\Pr_{w \leftarrow \mathsf{Ext}^{\mathbf{P}^*}(x)}[w \text{ is a witness for } x] \geq \Pr[1 \leftarrow \langle \mathsf{P}^*, \mathsf{V} \rangle(x)] - \epsilon_k$$

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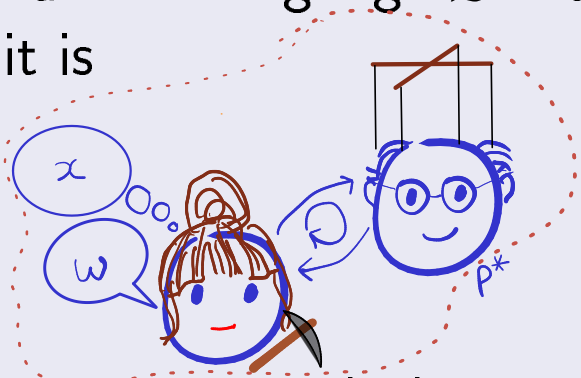

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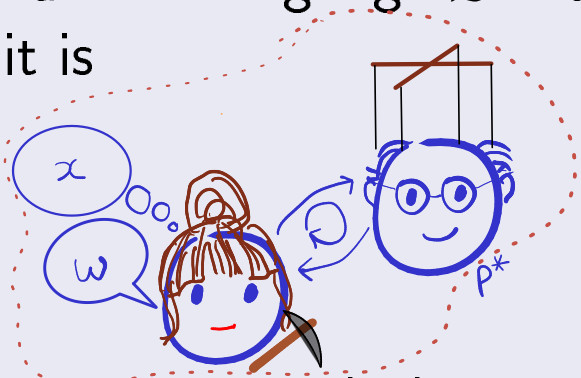

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- Trivial if we omit either of requirement 2 or 3

Ext must do something *more* than V, e.g. "rewind" P^*

Π_{GI} is ZKPoK: How to Extract π?

Theorem 5

Π_{GI} is a ZKPoK for $\mathcal{L}_{GI}$ with $\epsilon_k \leq 1/2$

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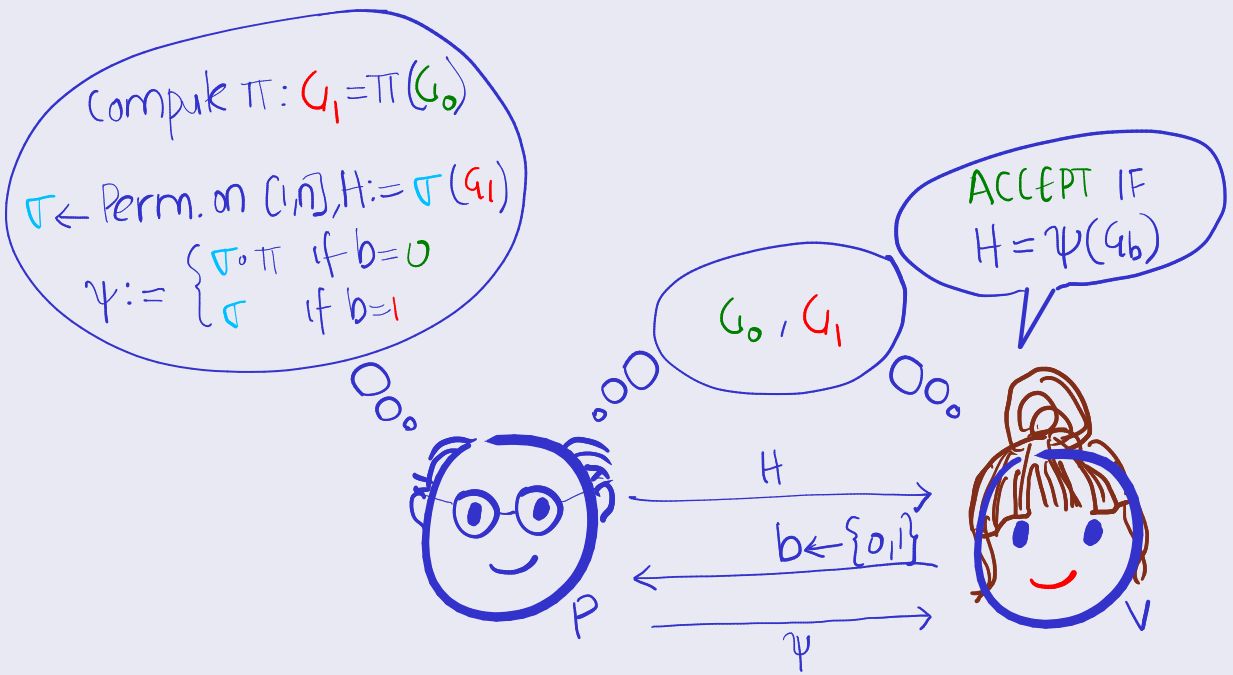
Proof (of PoK) Hint $\psi_1^{-1} \circ \psi_0 = \sigma^{-1} \circ \sigma \circ \pi = \pi$.



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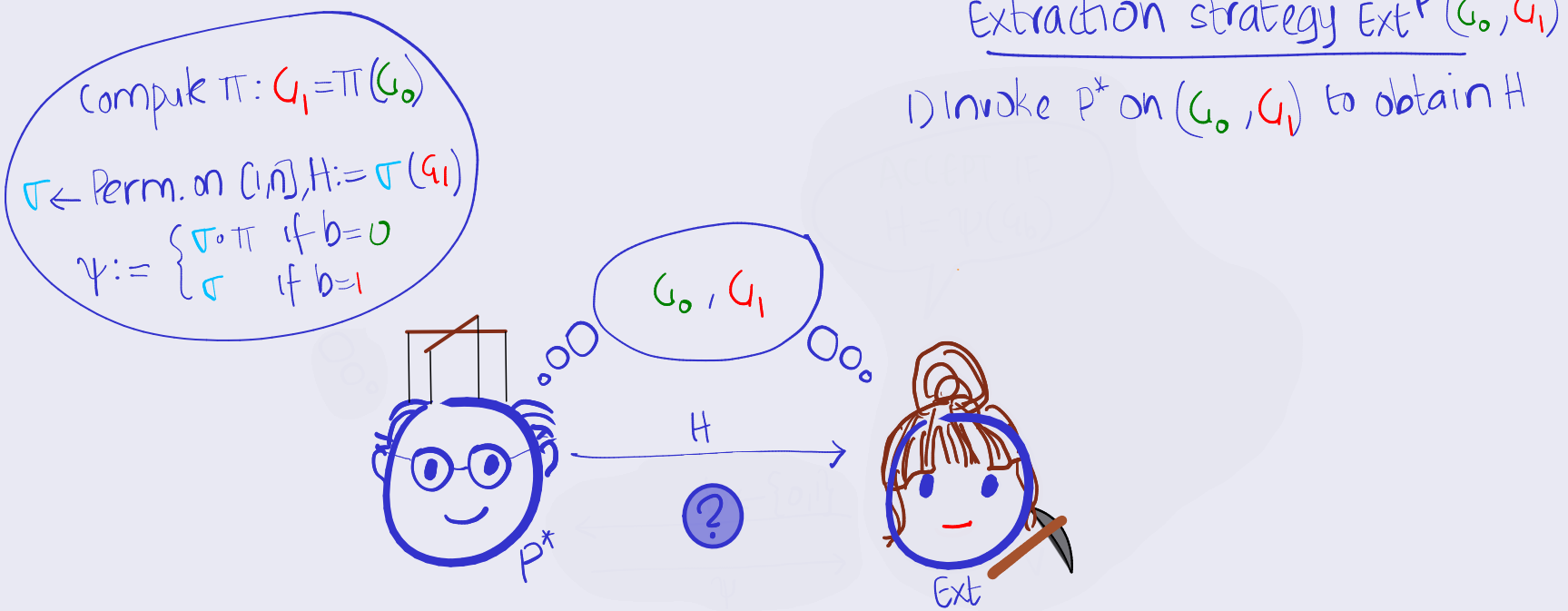

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Extraction strategy $\text{Ext}^{P^*}(G_0, G_1)$

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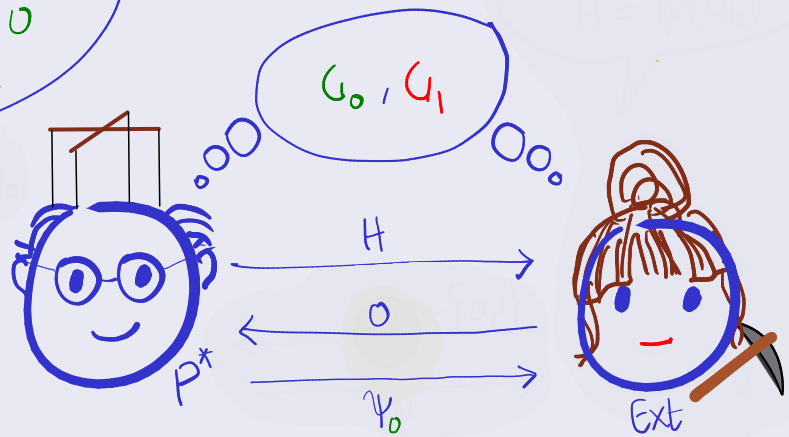

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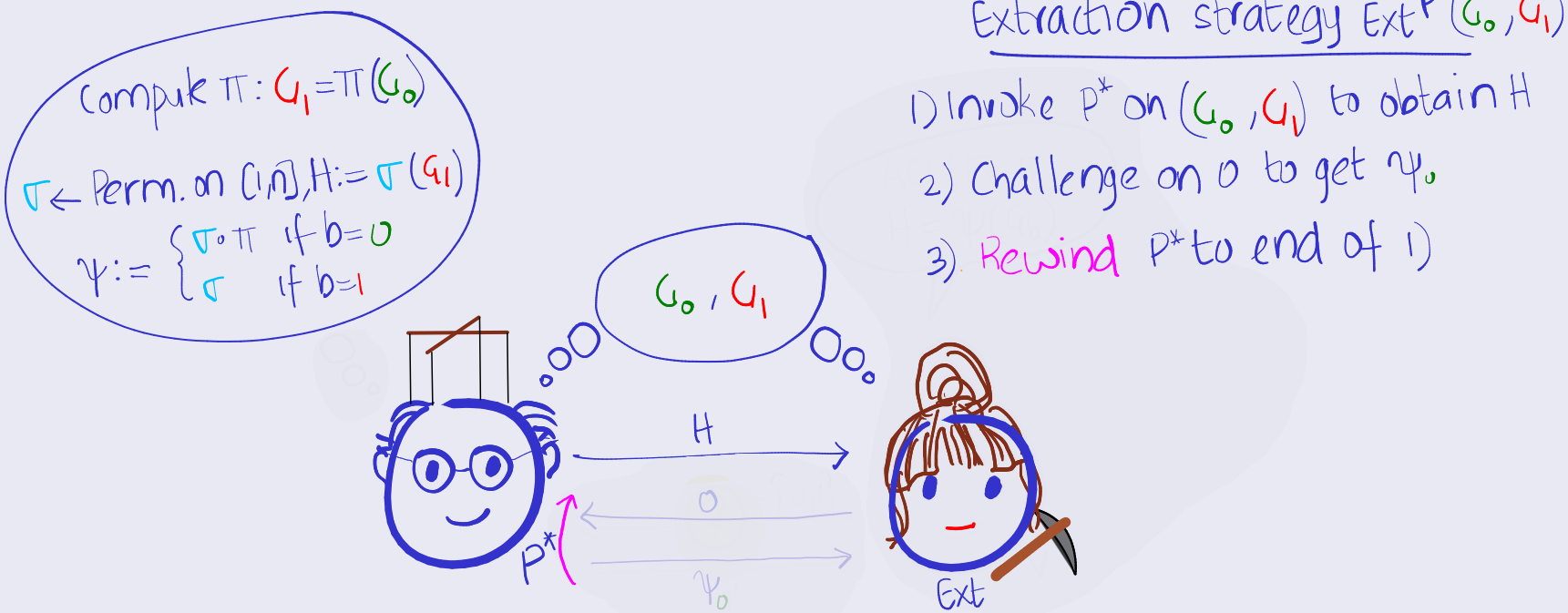

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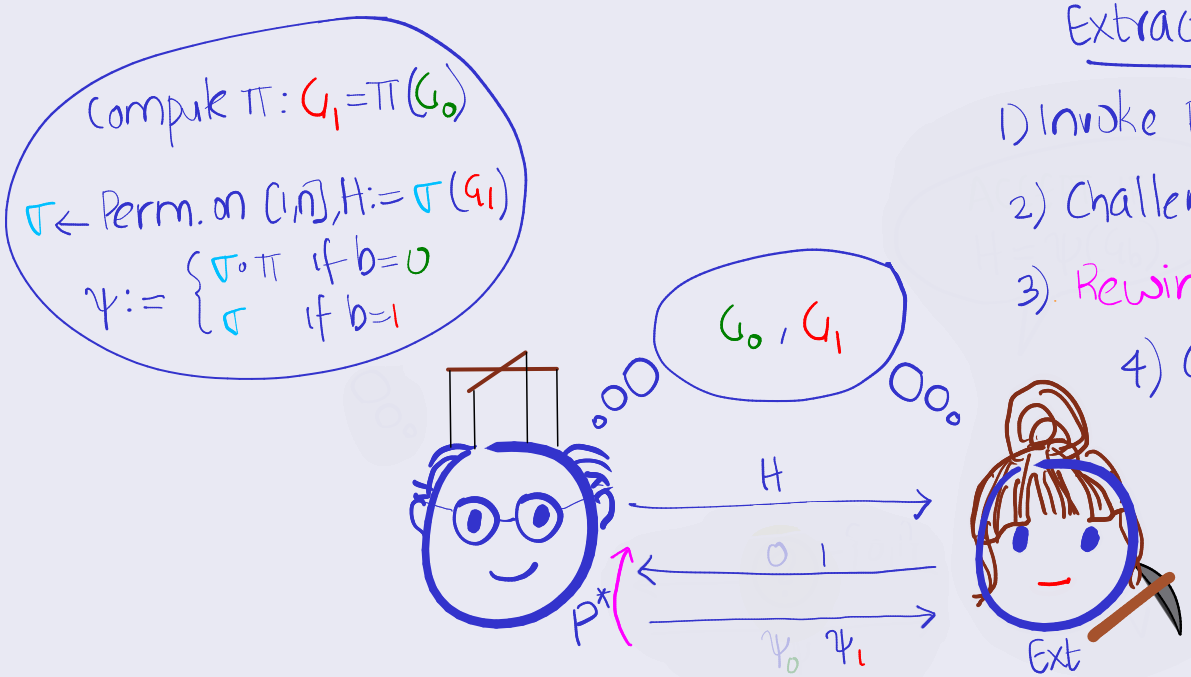

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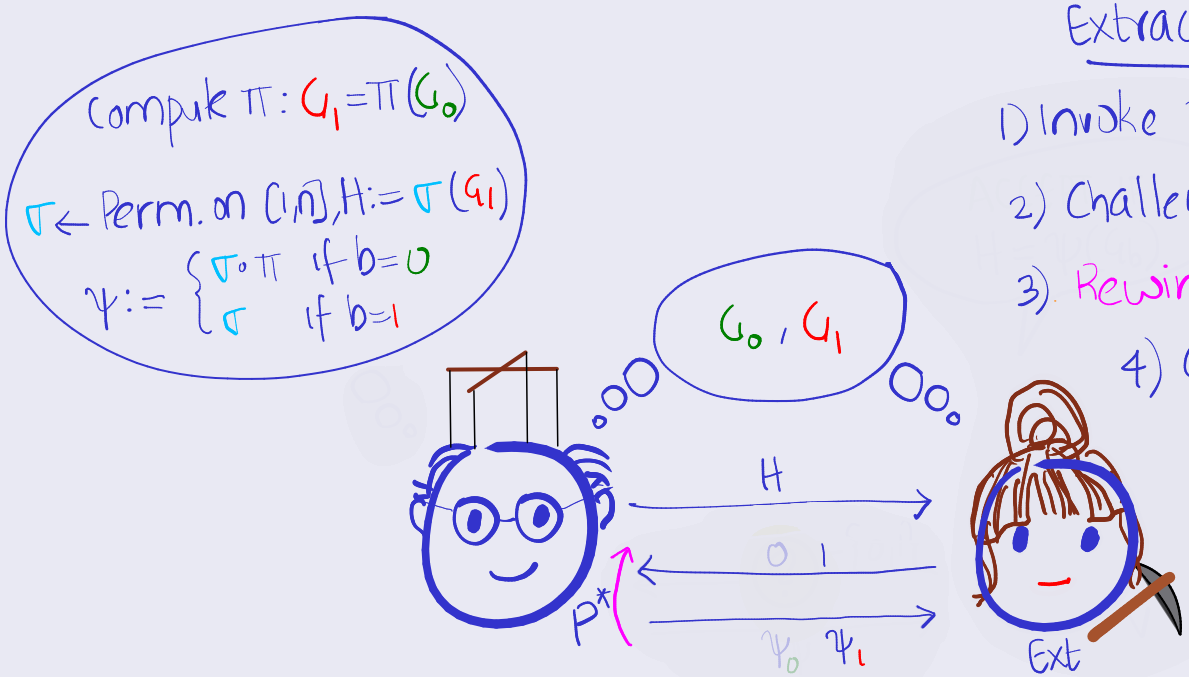

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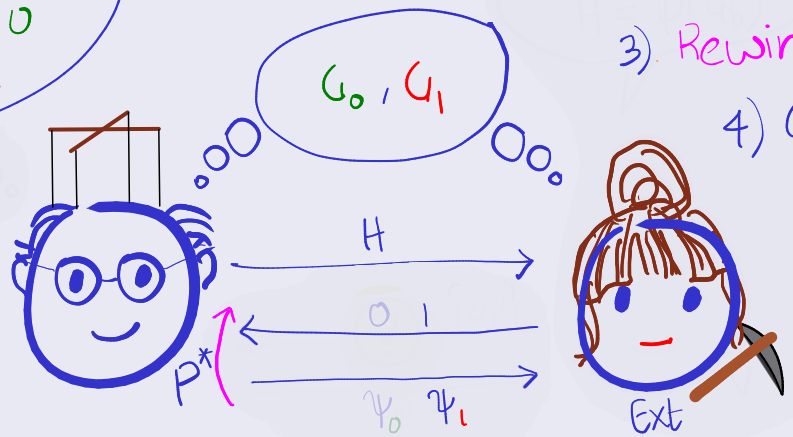

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V accepts 2) & 4) $\Rightarrow H = \psi_0(G_0) = \psi_1(G_1) \Rightarrow G_1 = \psi_1^{-1} \circ \psi_0(G_0)$ □

ZKPoK for DLP: Schnorr's Protocol...

Definition 3 (Lecture 12, DLP in prime-order $\mathbb{G}$ w.r.to S)

- Input:
  1. $(\mathbb{G}, \ell, g)$ sampled by a group sampler $\mathsf{S}(1^n)$
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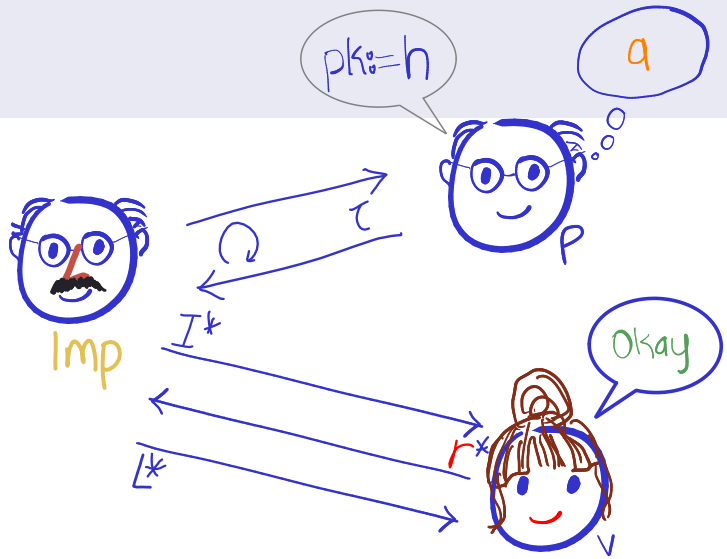
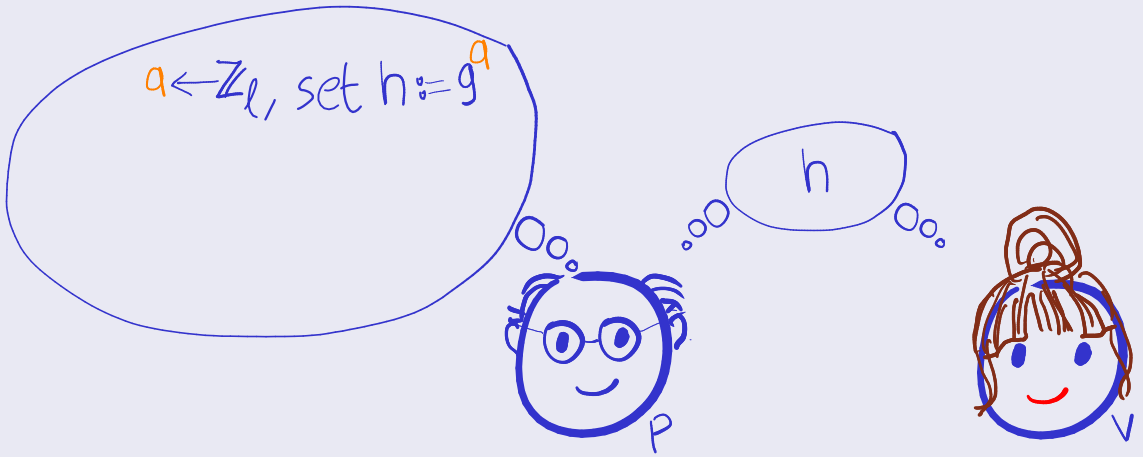


- ElGamal PKE:
  - Public key: $h := g^a$
  - Secret key: a
- In ID protocol for ElGamal PK, the impostor (who doesn't know a):
  - May see several transcripts via Auth_{sk} oracle
  - Should not be able to fool the verifier into accepting *in the protocol*

ZKPoK for DLog: Schnorr's Protocol...

Protocol 3 (Π_{DLP}: Schnorr's protocol)



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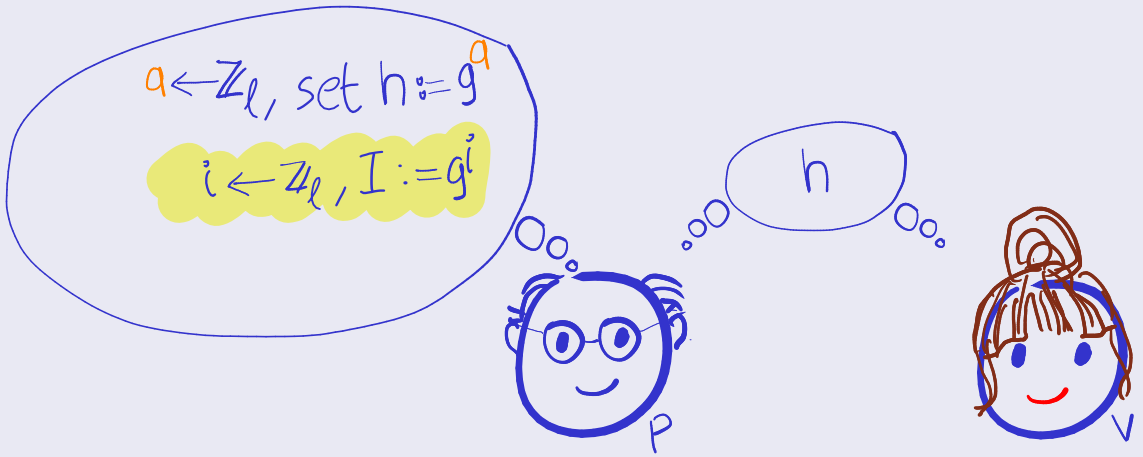

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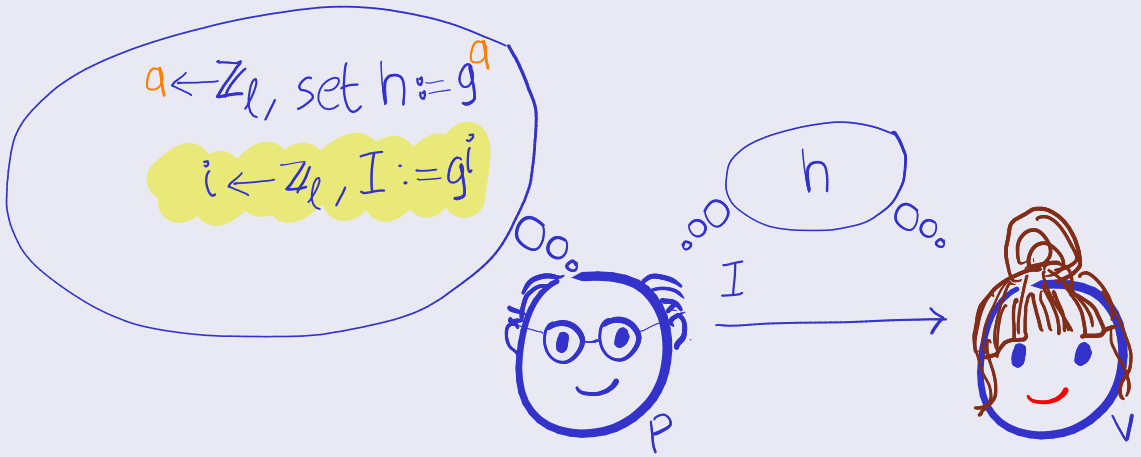

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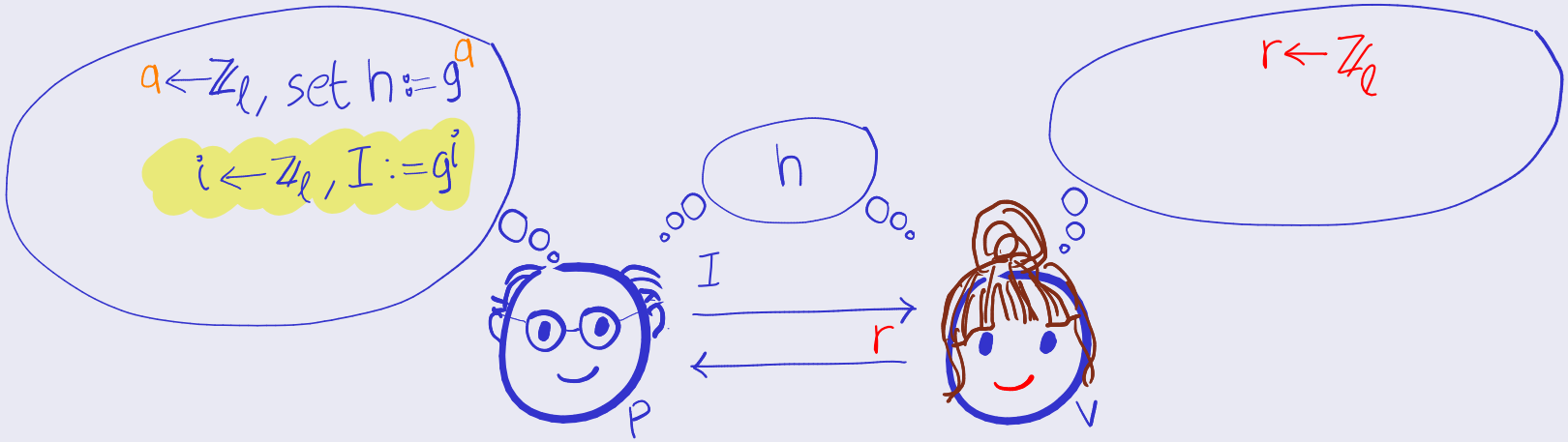

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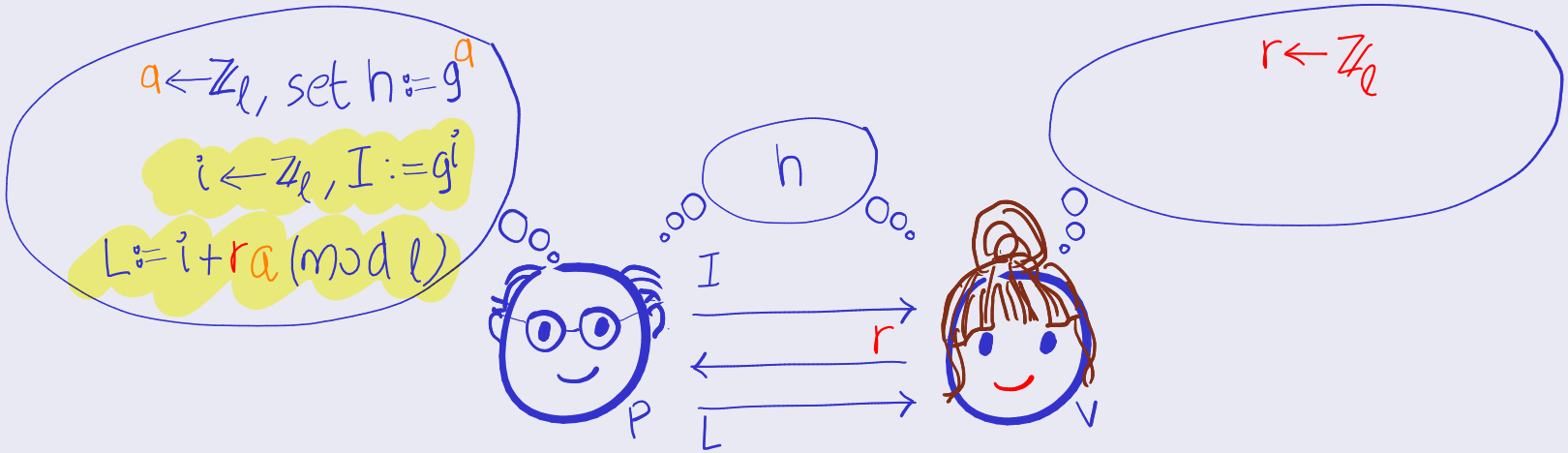

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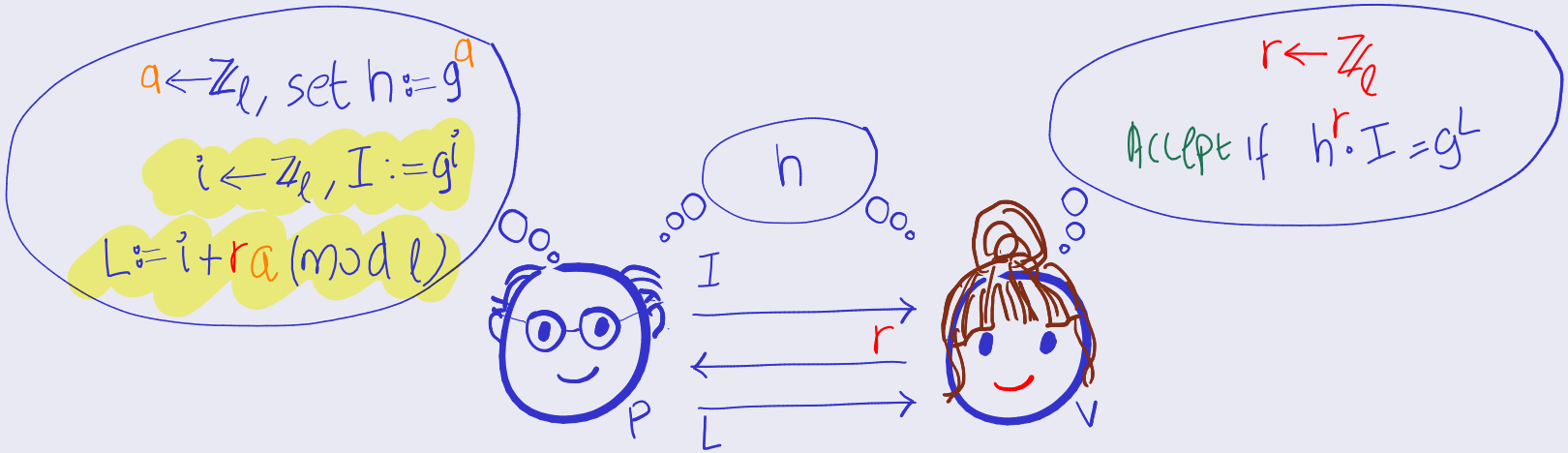

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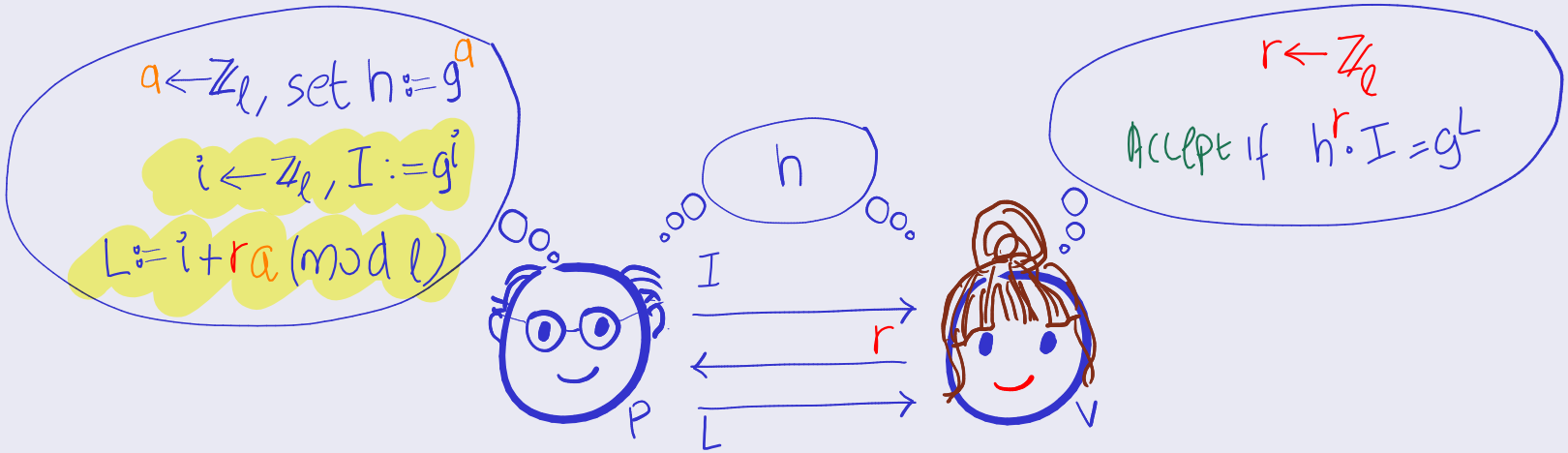

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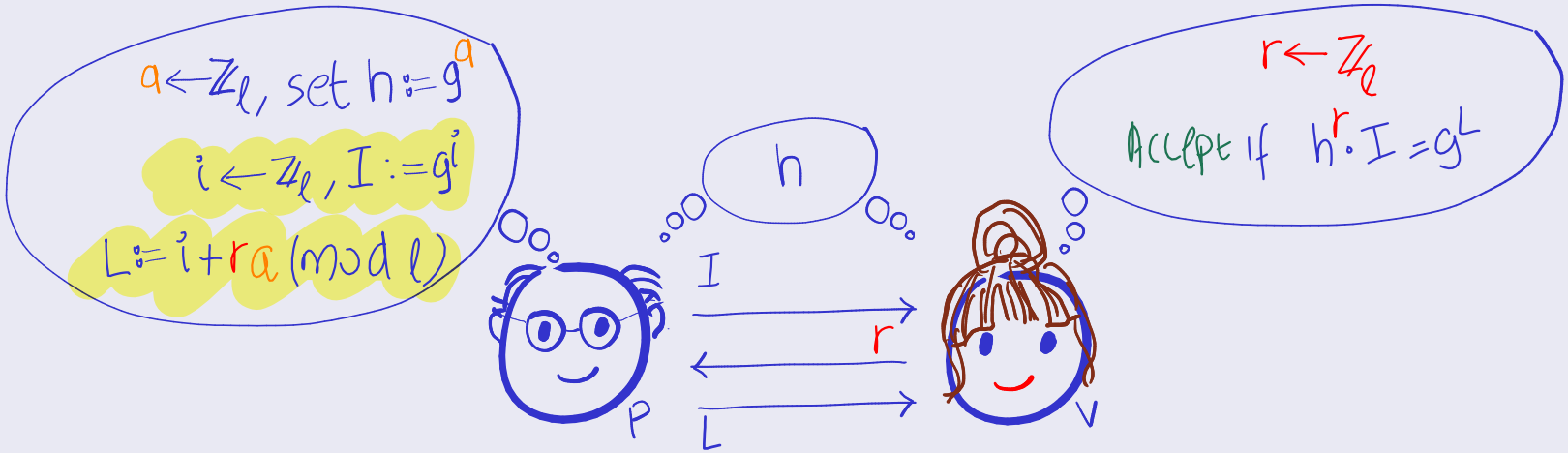

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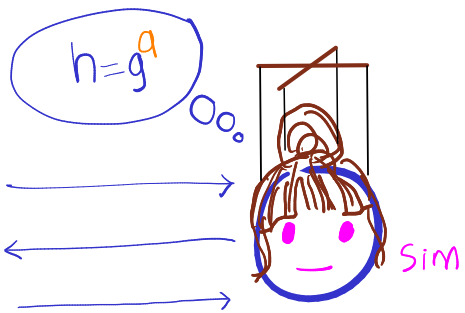

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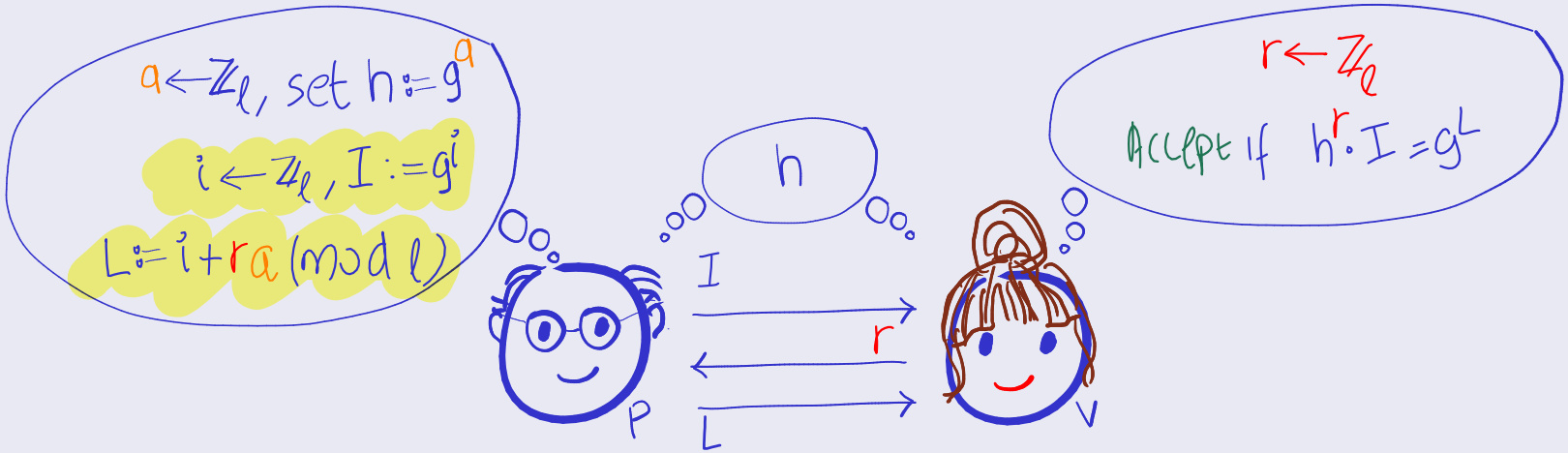

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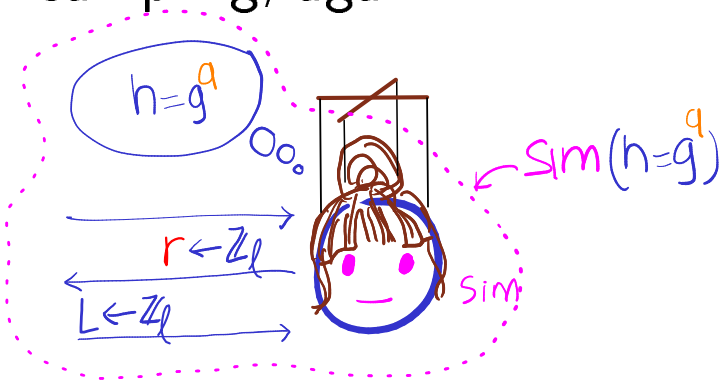

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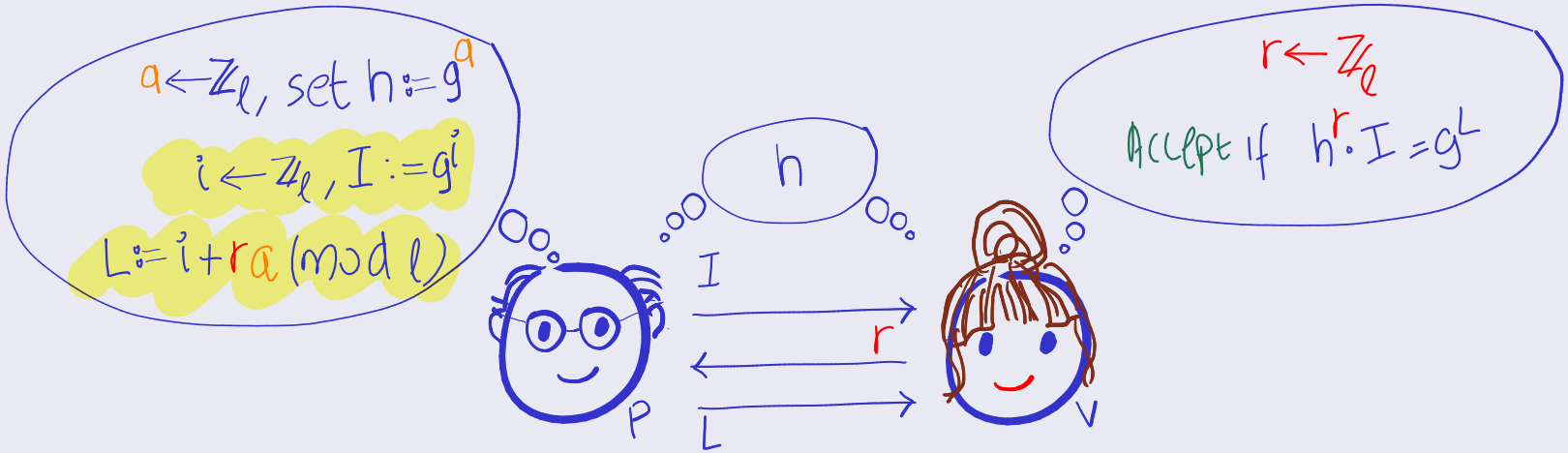

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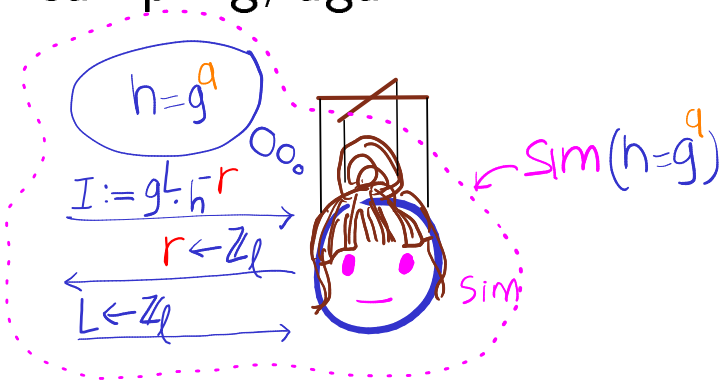

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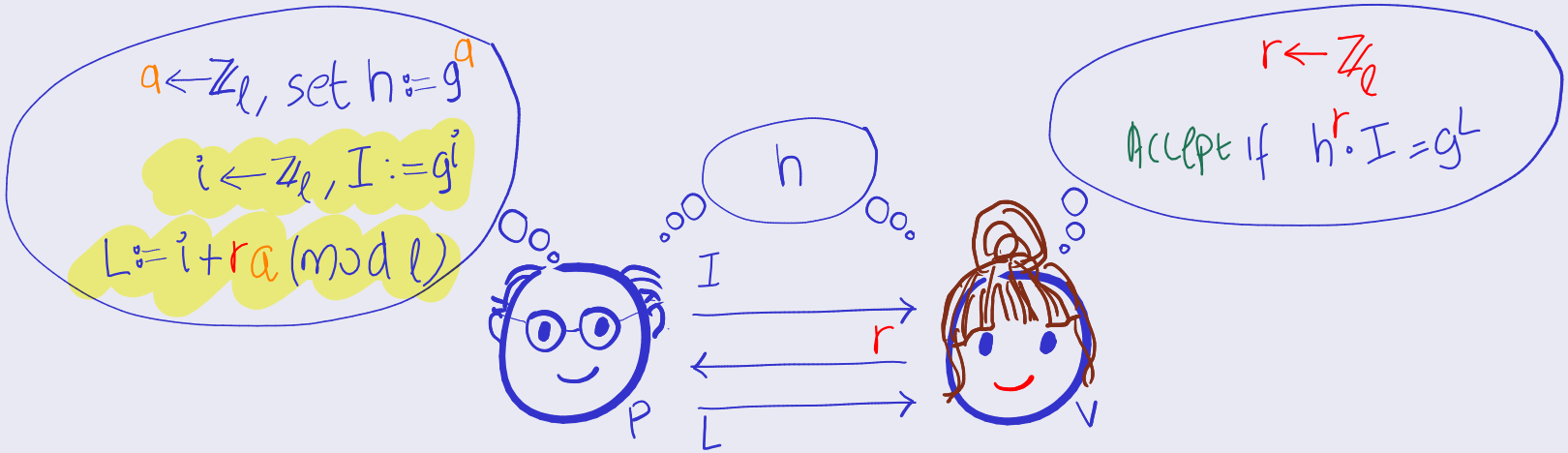

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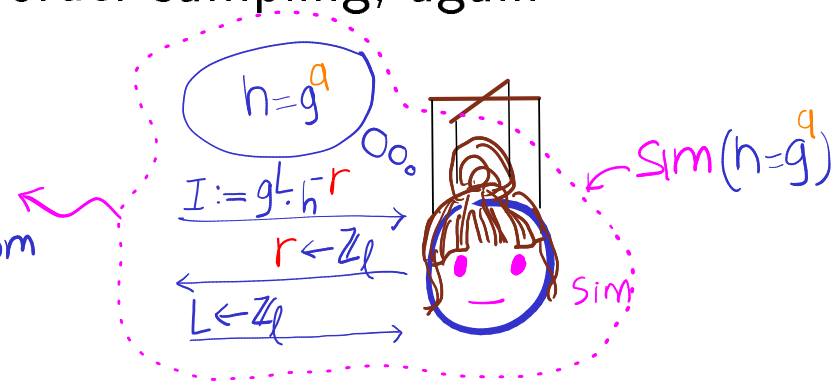
Distributed identically to view_V since

$g^L \cdot h^{-r} = g^{L-ar}$ is random



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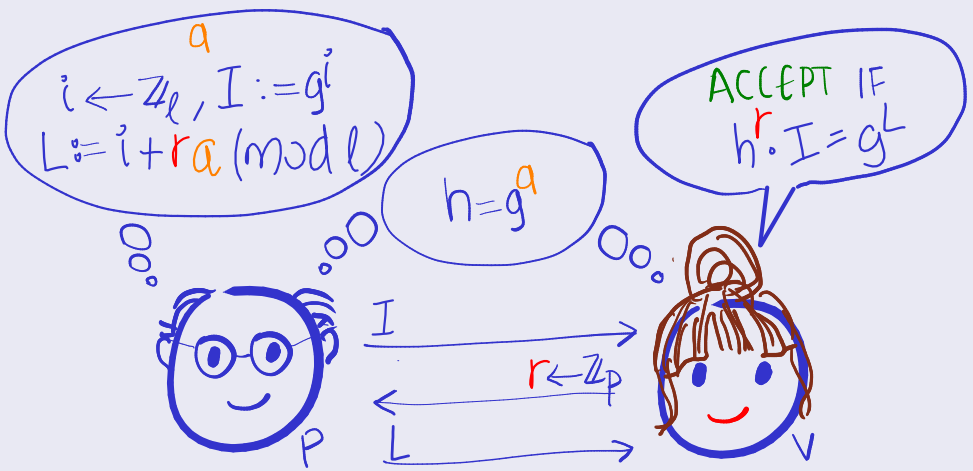
How to Extract a from P^*?

Theorem 6

Π_{DLP} *is a PoK for* $\mathcal{L}_{DLP}$ *with* $\epsilon_k \leq 1/p$

Proof (of PoK) Hint Obtain two eqns of form $L = I + ra \bmod \ell$.



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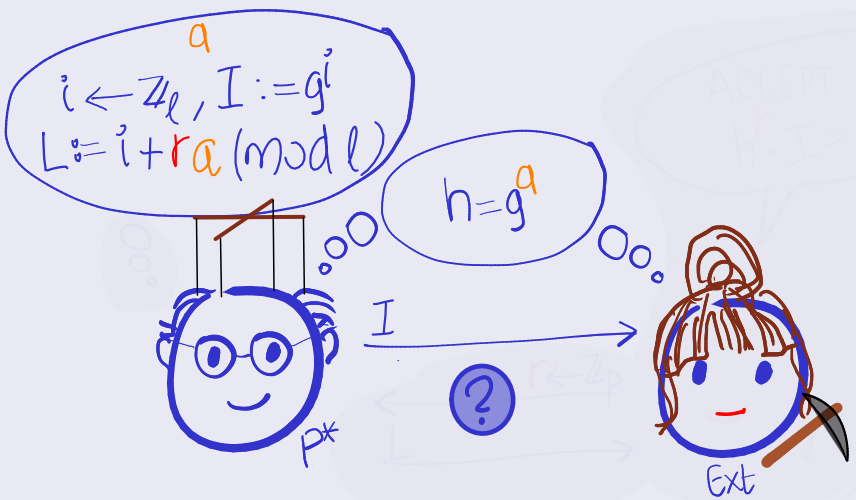

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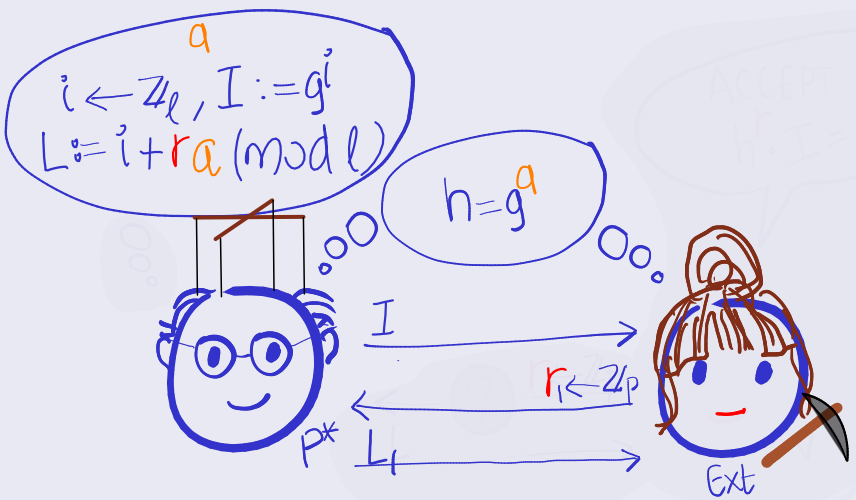

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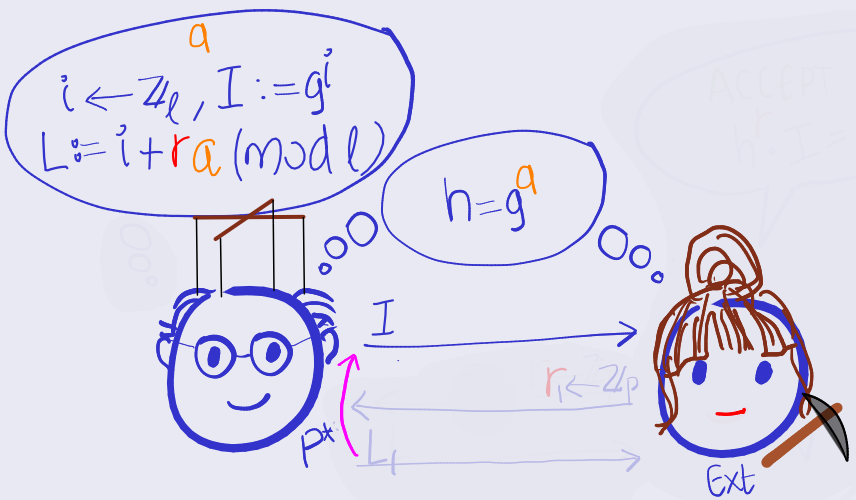

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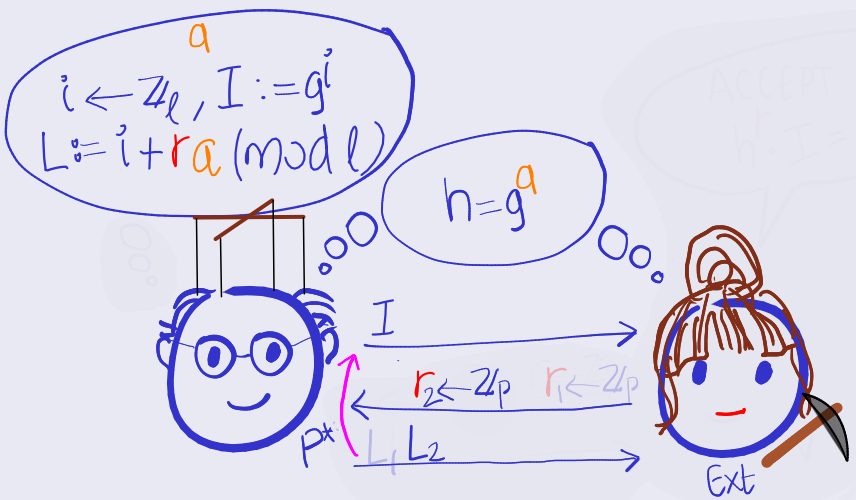

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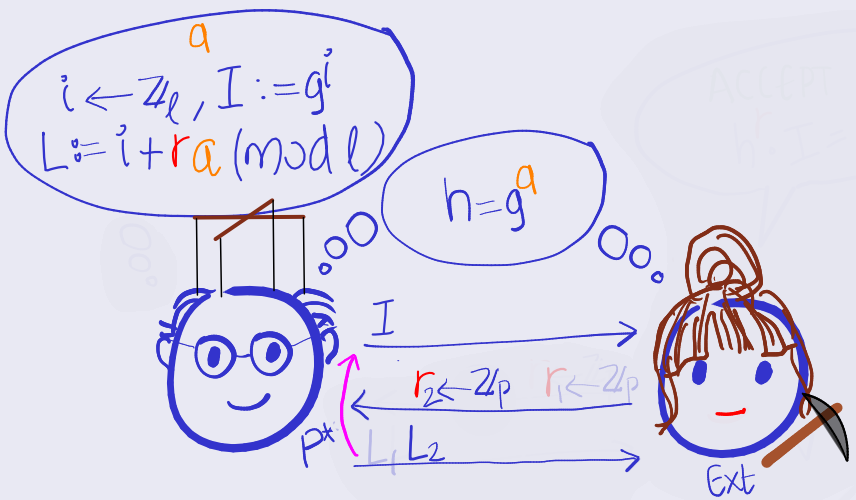

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5) Output $L_1 - L_2 / r_1 - r_2$

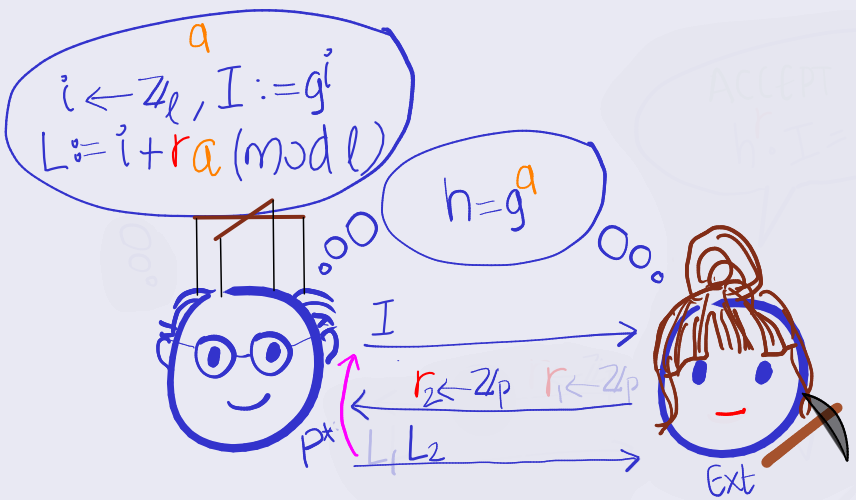
V accepts both executions $\Rightarrow$ $g^{L_1} = I \cdot h^{r_1}$ & $g^{L_2} = I \cdot h^{r_2}$ $\Rightarrow$ $g^{L_1 - L_2} = h^{r_1 - r_2}$ $\Rightarrow$ $a = \frac{L_1 - L_2}{r_1 - r_2}$

How to Extract a from P*?

Theorem 6

Π_{DLP} *is a PoK for* $\mathcal{L}_{DLP}$ *with* $\epsilon_k \leq 1/p$

Proof (of PoK) Hint Obtain two eqns of form $L = I + ra \bmod \ell$.



Extraction strategy $Ext^{P^*}(h)$

1) Invoke P^* on h to obtain I
2) Challenge on $r_1 \leftarrow Z_\ell$ to get L_1
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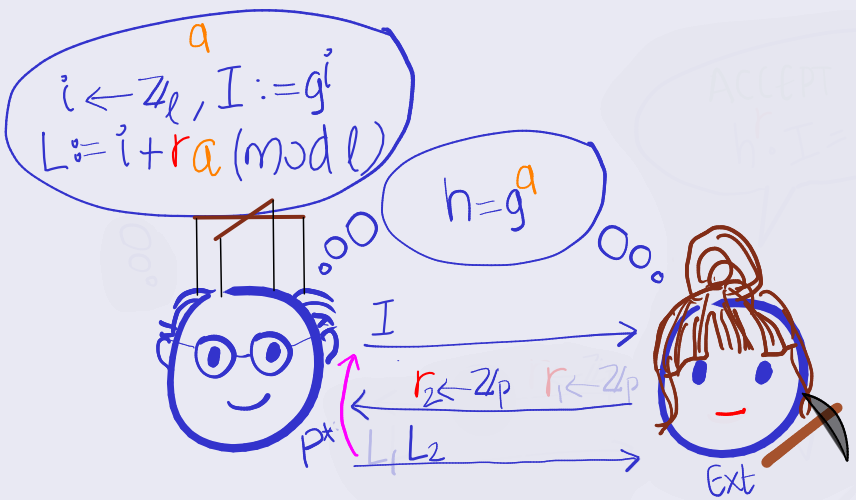

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- Next Lecture:
  - Application: eVoting
  - Tools used: Elgamal PKE, ZK (PoK)

References

1. [Gol01, Chapters 4.3 and 4.7] for details of today's lecture
2. [GMR89] for definitional and philosophical discussion on ZK
3. The ZKPs for GI and GNI are taken from [GMR89, GMW91]
4. Computational ZKP for all of PSPACE is due to [GMW91].


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