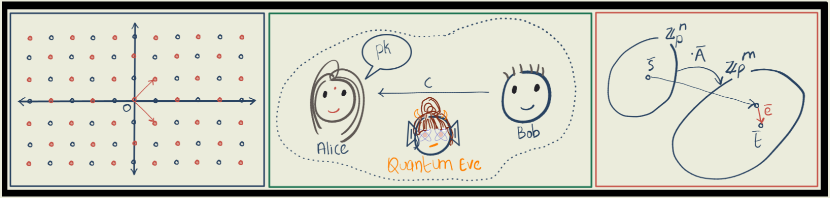


CS758: Advanced Tools from Modern Cryptography

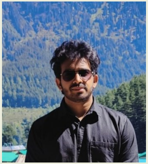
Lattice-Based Cryptography

Lecture 01 (06/Jan/26)

Instructor: Chethan Kamath

Administrivia..

- When and where: Slot 10 (14:00-15:25, Tuesdays and Fridays), CC101
- Contact hours: drop by my office (CC305) any time!
- Teaching assistant: Priyanshu Singh (24M2101)



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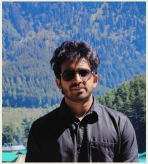

- Announcements/online discussion on Moodle:
  - moodle.iitb.ac.in/course/view.php?id=9123
- Any volunteers for class rep?
  - Please set up WhatsApp/Signal group for coordination

Administrivia..

- Grading Scheme

Weightage	Towards
30%	End-sem
25%	Mid-sem
20%	Paper presentation (two students per paper)
15%	Two quizzes
5%	Class participation
5%	Scribing (1-2 lectures per student)

Administrivia..

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30% +5	End-sem
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~~20%~~	~~Paper presentation (two students per paper)~~
15% +10	Two quizzes
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  - Please submit scribe notes within two weeks
  - Will scrap paper presentation and readjust grades if strength > 20
  - Attendance is not mandatory (but encouraged)

Administrivia..

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  - cse.iitb.ac.in/∼ckamath/courses/2026s/CS758.html

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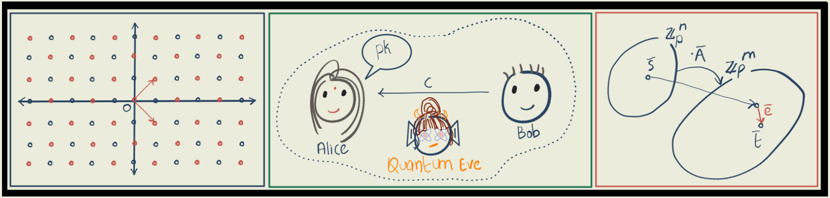
⚠ Pre-requisites

- ⚠ Will assume basic cryptography (CS409/CS409m/CS783)
- ☆ Brush up your linear algebra
- ☆ Useful to know basic coding theory and complexity theory
  - TA will conduct recitation sessions for these topics



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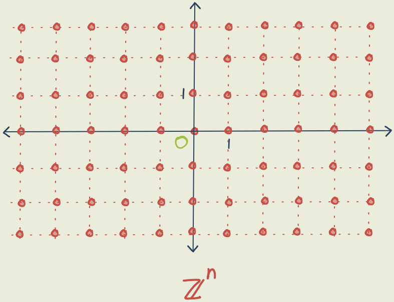
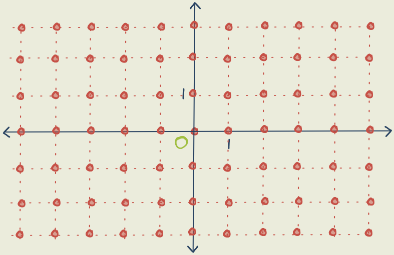
What are Lattices?



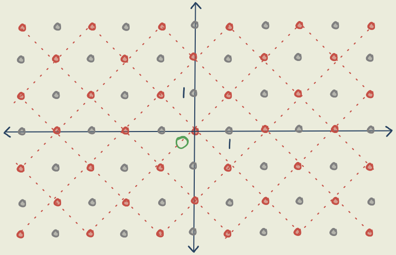
Informal Definition 1

Discrete subspace of (vector space) $\mathbb{R}^n$

What are Lattices?



$\mathbb{Z}^n$



$\{\text{integer vector } \bar{x} : \sum_i x_i \text{ is even}\}$

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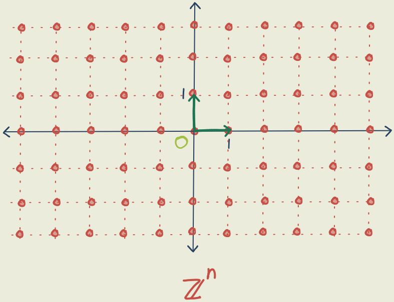

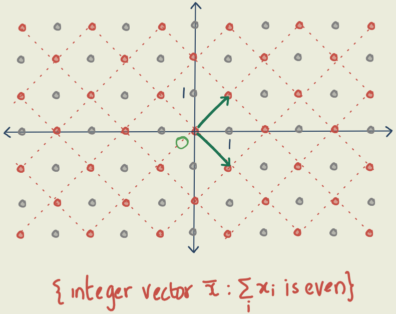

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Integer linear combinations of linearly-independent vectors $\bar{b}_1, \ldots, \bar{b}_m \in \mathbb{R}^n$

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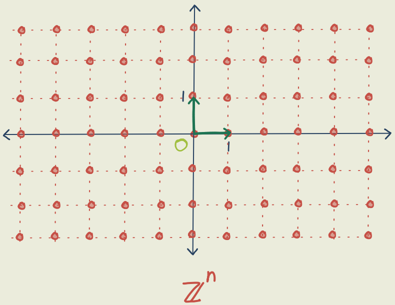

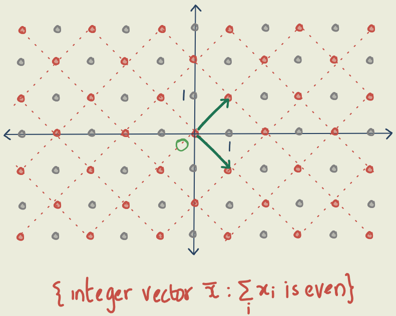

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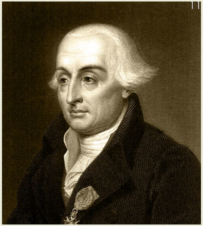
rank

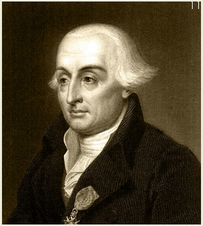
dimension

How are Lattices Useful?



- Historical applications
  1. Tool to study pure mathematics
     - E.g., as a bridge between number theory and geometry: can prove Fermat's theorem on sums of two squares using lattices

How are Lattices Useful?



Lagrange

Gauß

Minkowski

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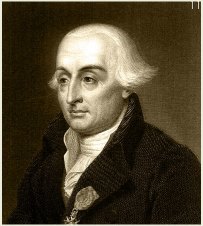
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- Historical applications
  1. Tool to study pure mathematics
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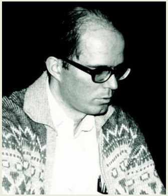
  2. Chemistry: to study crystal structure (crystallography)

  ...

How are Lattices Useful?..

Coppersmith

Lagarias



Odłyżko

- Modern applications
  1. Coding theory
  2. Cryptanalysis: breaking schemes using lattice algorithms

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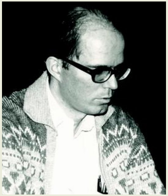
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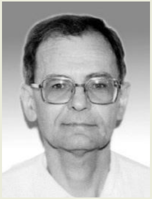
Ajtai


Regev

- Modern applications
  1. Coding theory
  2. Cryptanalysis: breaking schemes using lattice algorithms
  3. Lattice-based cryptography: constructing cryptographic primitives assuming hardness of lattice problems

  ...

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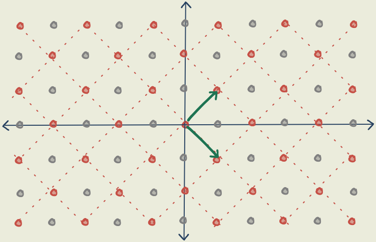

- Why lattice-based cryptography?
    1. Fast (linear) operations ⚡

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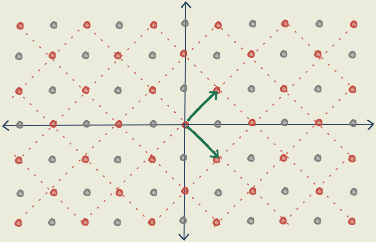

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        $\Rightarrow$ Cryptography based on worst-case assumptions!

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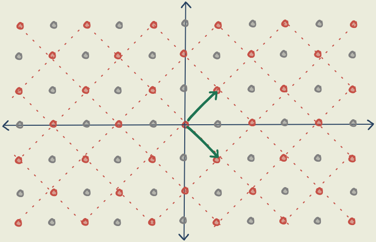

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  3. No efficient quantum algorithms known

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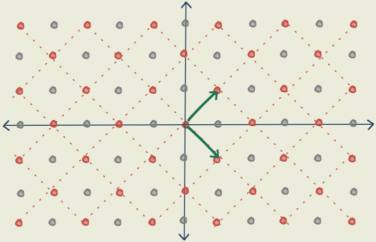

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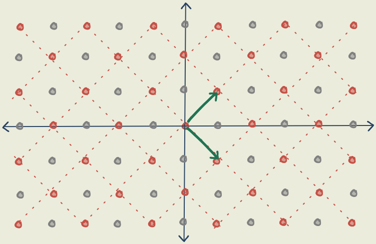
     $\Rightarrow$ Cryptography based on worst-case assumptions!
  3. No efficient quantum algorithms known

     $\Rightarrow$ Post-quantum cryptography
  4. Expressive! We know how to construct certain primitives only from lattices
     - E.g., Fully-Homomorphic Encryption (FHE)

CS758: Course Overview



1. Module I: Introduction to Lattices
2. Module II: Basic Cryptography from Lattices
3. Module III: Advanced Cryptography from Lattices

Module I: Introduction to Lattices.

- Study computational problems on lattices

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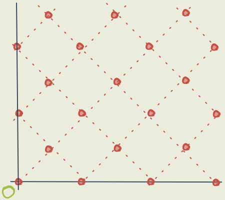
Shortest Vector Problem (SVP)

- I/p: lattice Λ



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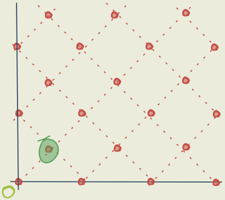

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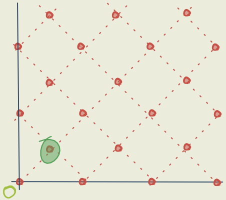

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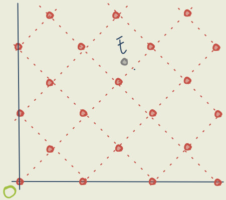
- I/p: lattice Λ, target $\bar{t} \in \mathbb{R}^n$



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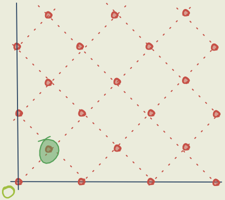

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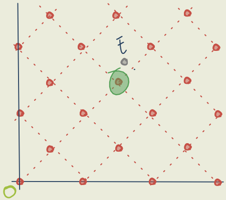

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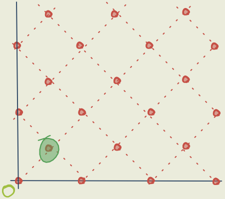

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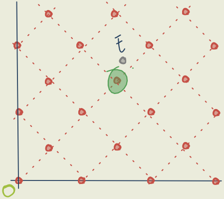

- Solution: lattice vector $\bar{x}$ *closest* to $\bar{t}$

- Also look at *estimation* and *approximation* variants
- Most problems hard to solve in the *worst case* ($\mathbf{NP}$-hard, in fact)

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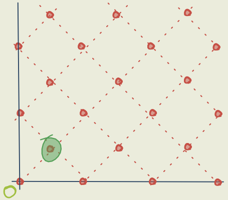

- Algorithms for lattice problems, e.g., LLL algorithm

Informal Theorem 1 ([LLL82])

There exists a polynomial-time algorithm that solves SVP up to an *exponential* (in n) approximation factor

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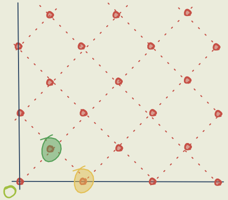

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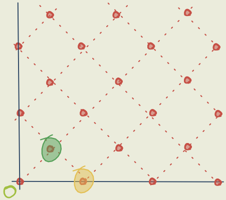

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Cryptanalysis using lattice algorithms:

- Factorise integers
- Solve low-exponent RSA
- Break Linear Congruential Generators (LCG)

Module II: Basic Cryptography from Lattices..

- Cryptography requires *average-case* hard problems
    - E.g., integer factorisation, discrete-logarithm problem (DLP)

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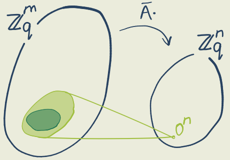
Short Integer Solution (SIS)

- I/p: matrix $\overline{A} \leftarrow \mathbb{Z}_q^{n \times m}$



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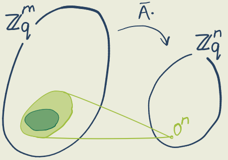

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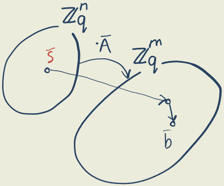
Learning With Errors (LWE)

- I/p: matrix $\overline{A} \leftarrow \mathbb{Z}_q^{n\times m}$ and $\bar{b}^t \approx \bar{s}^t\overline{A}$, where $\bar{s} \leftarrow \mathbb{Z}_q^n$



- Solution: $\bar{s}$

Module II: Basic Cryptography from Lattices..

- What is the connection to lattices?

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- What is the connection to lattices? Can be viewed as *average-case* variants of CVP

Informal Theorem 2 ([Ajt96] (resp., [Reg05]))

SIVP (resp., BDD) is worst-case hard $\Rightarrow$ SIS (resp., LWE) is average-case hard

Module II: Basic Cryptography from Lattices..

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Informal Theorem 2 ([Ajt96] (resp., [Reg05]))

SIVP (resp., BDD) is worst-case hard $\Rightarrow$ SIS (resp., LWE) is average-case hard

- What can we construct assuming SIS and LWE?

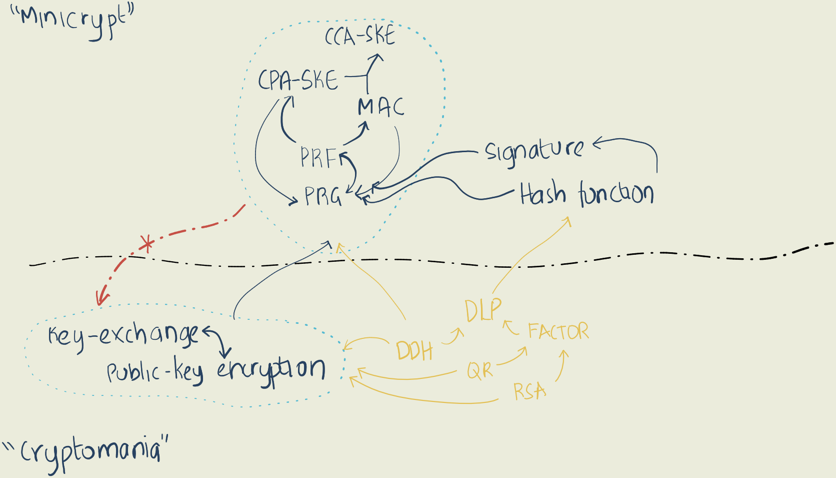
Informal Theorem 3 ([Ajt96])

SIS $\Rightarrow$ one-way function (OWF) and collision-resistant hash function

Informal Theorem 4 ([Reg05])

LWE $\Rightarrow$ public-key encryption (PKE)

The Cryptographic Landscape



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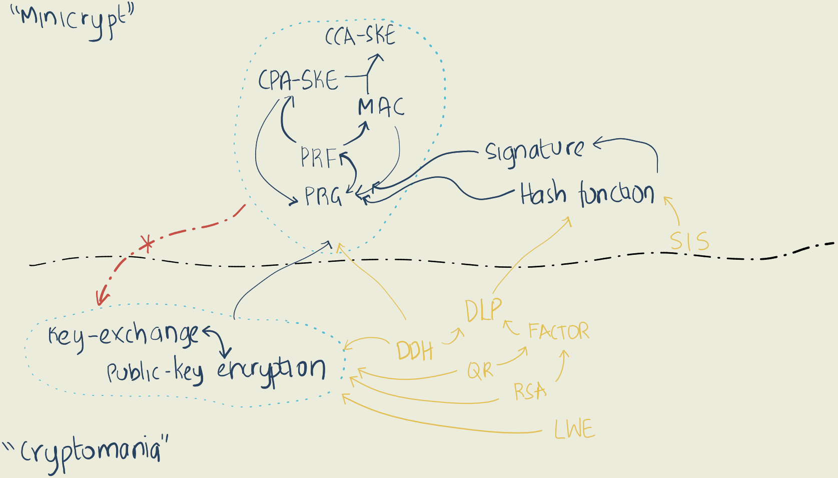

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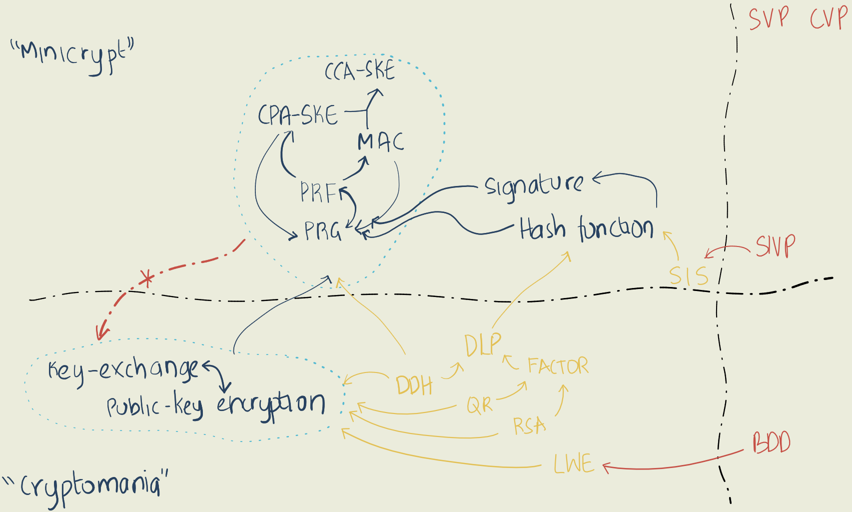

Module III: Advanced Cryptography from Lattices..

Advanced Primitive I: Identity-Based Encryption (IBE) [Sha84]

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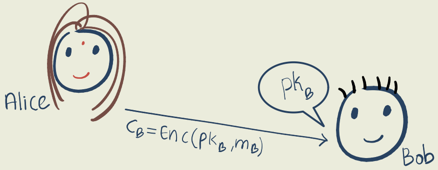
- Recall PKE (Public-Key Encryption)



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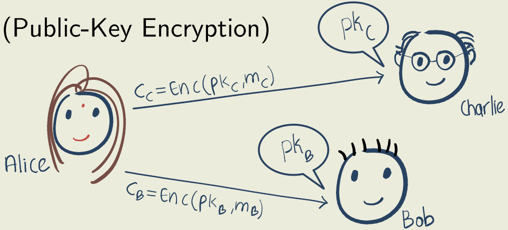

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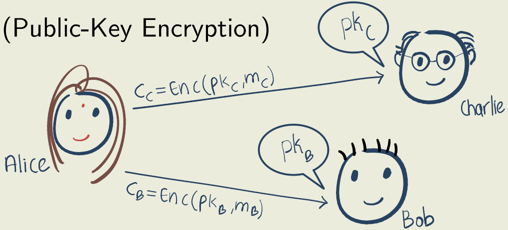

Drawback: need to maintain (authenticated) directory with identity (e.g., e-mail address) and PK pairs

Module III: Advanced Cryptography from Lattices..

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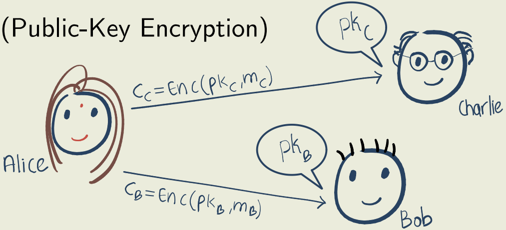

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IBE solves this issue: identity *itself* is the public key!

Informal Theorem 5 ([GPV08])

$\mathrm{LWE} \Rightarrow$ IBE

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Advanced Primitive I: Identity-Based Encryption (IBE) [Sha84]

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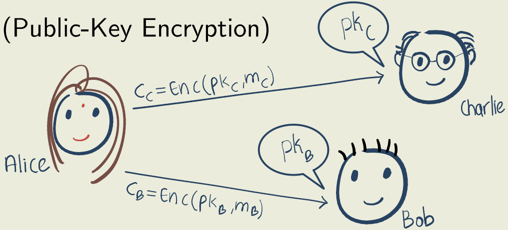


Drawback: need to maintain (authenticated) directory with identity (e.g., e-mail address) and PK pairs

IBE solves this issue: identity *itself* is the public key!

Informal Theorem 5 ([GPV08])

$\text{LWE} \Rightarrow$ IBE

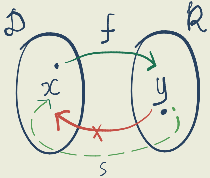
++ Generalisation of IBE: attribute-based encryption (ABE) [SW04]

- Application: secure access control (e.g., Cloudflare's GeoV2)

Module III: Advanced Cryptography from Lattices..

Advanced Primitive I: Identity-Based Encryption (IBE) [Sha84]

- Recall trapdoor functions: OWF that is easy to invert given a "secret" s



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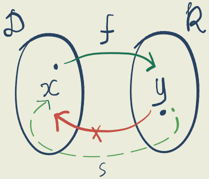

Key tool: lattice trapdoors, e.g., a "short" basis

Informal Theorem 6 ([Ajt96; MP12])

There exists a PPT algorithm that samples a SIS matrix $\overline{A} \leftarrow \mathbb{Z}_q^{n\times m}$ along with a "short" matrix $\overline{S} \in \mathbb{Z}^{m\times m}$ such that $\overline{A}\,\overline{S} = 0^{n\times m} \bmod q$

Module III: Advanced Cryptography from Lattices..

Advanced Primitive II: FHE (Fully-Homomorphic Encryption) [RAD78]

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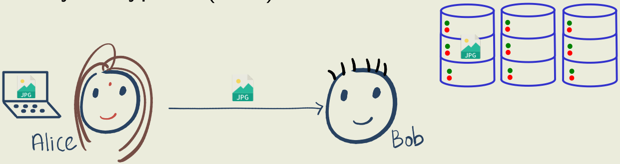
- Recall secret-key encryption (SKE)



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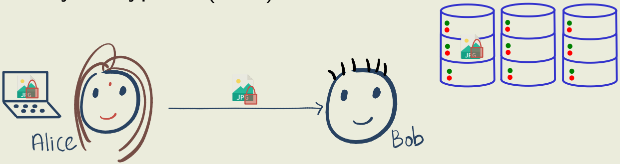

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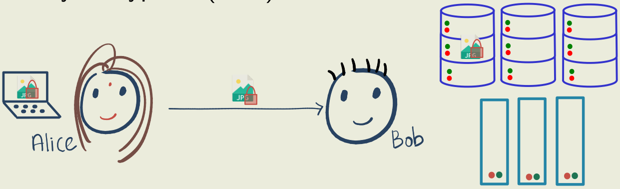

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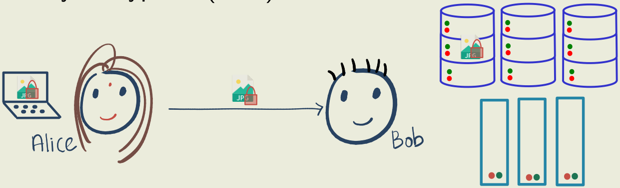

Drawback: no “fine-grained” access

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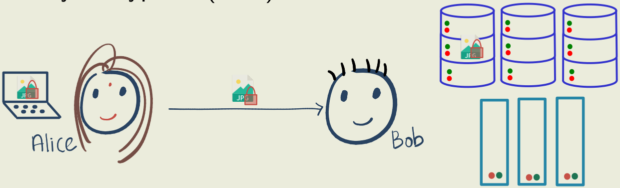

Drawback: no “fine-grained” access

FHE: allows computing *over the ciphertext*!

Informal Theorem 7 ([GSW13])

$\mathrm{LWE} \Rightarrow$ FH(PK)E

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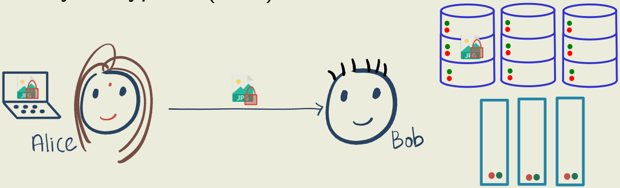

Drawback: no "fine-grained" access

FHE: allows computing *over the ciphertext*!

Informal Theorem 7 ([GSW13])

$\mathrm{LWE} \Rightarrow$ FH(PK)E

- Applications:
  - Private (zero-trust) outsourcing of computation
  - Encrypted search (e.g., see Zama)

Lecture Resources

1. *Lecture 1: Introduction* of CSE206A (Spring 2007), by Micciancio (PDF)
2. *Historical Talk on Lattice-Based Cryptography*, Micciancio (YouTube)
3. *Mathematics of Lattices*, Micciancio (YouTube)

References

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