A Generalized Disjunctive Paraconsistent Data Model

Haibin Wang and Rajshekhar Sunderraman

Department of Computer Science
Georgia State University
Atlanta, GA 30302, USA
email: hwang17@student.gsu.edu, raj@cs.gsu.edu

Abstract

This paper presents a generalization of the disjunctive paraconsistent relational data model in which disjunctive positive and negative information can be represented explicitly and manipulated. There are situations where the closed world assumption to infer negative facts is not valid or undesirable and there is a need to represent and reason with negation explicitly. We consider explicit disjunctive negation in the context of disjunctive databases as there is an interesting interplay between these two types of information. *Generalized disjunctive paraconsistent relation* is introduced as the main structure in this model. The relational algebra is appropriately generalized to work on generalized disjunctive paraconsistent relations and their correctness is established.

1 Introduction

Two important features of the relational data model [4] for databases are its value-oriented nature and its rich set of simple, but powerful algebraic operators. Moreover, a strong theoretical foundation for the model is provided by the classical first-order logic [15]. This combination of a respectable theoretical platform, ease of implementation and the practicality of the model resulted in its immediate success, and the model has enjoyed being used by many database management systems.

One limitation of the relational data model, however, is its lack of applicability to nonclassical situations. These are situations involving incomplete or even inconsistent information.

Several types of incomplete information have been extensively studied in the past such as *null* values [5, 10], *partial* values [8], *fuzzy* and *uncertain* values [7, 14], and *disjunctive* information [11, 12, 17]. Liu and Sunderraman in [11] and [12] presented extensions to relational data model that were capable of representing and manipulating disjunctive information.

However, unlike incomplete information, inconsistent information has not enjoyed enough research attention. While it may be argued that true knowledge systems should contain no inconsistent information, contradictions are very common in belief systems. Even experts of a domain often disgaree with each other, sometimes strongly. Logics dealing with inconsistent information are called *paraconsistent* logics, and were studied in detail by de Costa [6] and Belnap [2]. Blair and Subrahmanian [3] proposed logic programming based on paraconsistent logic and Subrahmanian [16] extended the work to disjunctive deductive databases. Bagai and Sunderraman [1] presented a paraconsistent relational data model to represent and manipulate explicit negation in relational databases. More recently, Grant and Subrahmanian [9] present a survey of work in paraconsistent databases and knowledgebases.

In this paper, we present an extension to the relational data model that is capable of representing and manipulating explicitly stated positive disjunctive facts as well as explicitly stated negative disjunctive facts. The model proposed in this paper is a generalization of the disjunctive paraconsistent data model[18]. We introduce *generalized disjunctive paraconsistent relations*, which are the fundamental structures underlying our model. These structures are generalizations of *disjunctive paraconsistent relations* which are capable of representing disjunctive positive and explicit negative definite facts. A generalized disjunctive paraconsistent relation essentially consists of two kinds of information: positive tuple sets representing exclusive disjunctive positive facts (one of which belongs to the relation) and negative tuple sets representing exclusive disjunctive negated facts (one of which does not belong to the relation). Generalized disjunctive paraconsistent relations are strictly more general than disjunctive paraconsistent relations in that for any disjunctive paraconsistent relation, there is a generalized disjunctive paraconsistent relation with the same information content, but not *vice*

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versa. We define algebraic operators over generalized disjunctive paraconsistent relations that extend the standard operations over disjunctive paraconsistent relations.

In Sections 2 and 3 we briefly present the data models that precede the generalization presented in this paper. Paraconsistent relations allow unit positive and negative facts to be represented. Disjunctive paraconsistent relations allow disjunctive positive facts and unit negative facts to be represented. In Section 4, we present the generalization that allows disjunctive positive as well as disjunctive negative facts to be represented. The relational algebra is then generalized in Section 5. The paper is concluded in Section 6 with directions for future work.

2 Paraconsistent Relations

In this section, we present a brief overview of paraconsistent relations and the algebraic operations on them. For a more detailed description, refer to [1]. Unlike ordinary relations that can model worlds in which every tuple is known to either hold a certain underlying predicate or to not hold it, paraconsistent relations provide a framework for incomplete or even inconsistent information about tuples. They naturally model *belief* systems rather than *knowledge* systems, and are thus a generalisation of ordinary relations. The operators on ordinary relations can also be generalised for paraconsistent relations.

Let a *relation scheme* (or just *scheme*) Σ be a finite set of *attribute names*, where for any attribute name $A \in \Sigma$, $dom(A)$ is a non-empty *domain* of values for A. A *tuple* on Σ is any map $t : \Sigma \to \cup_{A\in\Sigma} dom(A)$, such that $t(A) \in dom(A)$, for each $A \in \Sigma$. Let $\tau(\Sigma)$ denote the set of all tuples on Σ.

Definition 1 *A* paraconsistent relation *on scheme* Σ *is a pair* $R = \langle R^+, R^-\rangle$, *where* R^+ *and* R^- *are any subsets of* $\tau(\Sigma)$. *We let* $\mathcal{P}(\Sigma)$ *be the set of all paraconsistent relations on* Σ. □

Intuitively, R^+ may be considered as the set of all tuples for which R is believed to be true, and R^- the set of all tuples for which R is believed to be false. Note that since contradictory beliefs are possible, we do not assume R^+ and R^- to be mutually disjoint, though this condition holds in an important class of paraconsistent relations. As paraconsistent relations may contain contradictory information, they model belief systems more naturally than knowledge systems. Furthermore, R^+ and R^- may not together cover all tuples in $\tau(\Sigma)$.

Definition 2 *A paraconsistent relation* R *on scheme* Σ *is called a* consistent paraconsistent relation *if* $R^+ \cap R^- = \emptyset$. *We let* $\mathcal{C}(\Sigma)$ *be the set of all consistent relations on* Σ. *Moreover,* R *is called a* complete paraconsistent relation *if* $R^+ \cup R^- = \tau(\Sigma)$. *If* R *is both consistent and complete, i.e.* $R^- = \tau(\Sigma) - R^+$, *then it is a* total paraconsistent relation, *and we let* $\mathcal{T}(\Sigma)$ *be the set of all total paraconsistent relations on* Σ. □

Algebraic Operators

To reflect generalization of relational algebra operators, a dot is placed over an ordinary relational operator to obtain the corresponding paraconsistent relation operator. For example, $\bowtie$ denotes the natural join among ordinary relations, and $\dot{\bowtie}$ denotes natural join on paraconsistent relations. We first introduce two fundamental set-theoretic algebraic operators on paraconsistent relations:

Definition 3 *Let* R *and* S *be paraconsistent relations on scheme* Σ. *Then, the* union *of* R *and* S, *denoted* $R \mathbin{\dot\cup} S$, *is a paraconsistent relation on scheme* Σ, *given by* $(R \mathbin{\dot\cup} S)^+ = R^+ \cup S^+$ *and* $(R \mathbin{\dot\cup} S)^- = R^- \cap S^-$; *and the* complement *of* R, *denoted* $\dot{-}\, R$, *is a paraconsistent relation on scheme* Σ, *given by* $(\dot{-}\, R)^+ = R^-$ *and* $(\dot{-}\, R)^- = R^+$. □

An intuitive appreciation of the union operator may be obtained by interpreting relations as properties of tuples. So, $R \mathbin{\dot\cup} S$ is the "either-R-or-S" property. Now since R^+ and S^+ are the sets of tuples for which the properties R and S, respectively, are believed to hold, the set of tuples for which the property "either-R-or-S" is believed to hold is clearly $R^+ \cup S^+$. Moreover, since R^- and S^- are the sets of tuples for which properties R and S, respectively, are believed to *not* hold, the set of tuples for which the property "either-R-or-S" is believed to *not* hold is similarly $R^- \cap S^-$.

The definition of *complement* and of all the other operators on paraconsistent relations defined later can (and should) be understood in the same way.

If Σ and Δ are relation schemes such that $\Sigma \subseteq \Delta$, then for any tuple $t \in \tau(\Sigma)$, we let t^Δ denote the set $\{t' \in \tau(\Delta) \mid t'(A) = t(A), \text{ for all } A \in \Sigma\}$ of all extensions of t. We extend this notion for any $T \subseteq \tau(\Sigma)$ by defining $T^\Delta = \cup_{t\in T}\, t^\Delta$. We now define some relation-theoretic operators on paraconsistent relations.

Definition 4 *Let* R *and* S *be paraconsistent relations on schemes* Σ *and* Δ, *respectively. Then, the* natural join *(or just* join*) of* R *and* S, *denoted* $R \mathbin{\dot\bowtie} S$, *is a paraconsistent relation on scheme* $\Sigma \cup \Delta$, *given by* $(R \mathbin{\dot\bowtie} S)^+ = R^+ \bowtie S^+$, *and* $(R \mathbin{\dot\bowtie} S)^- = (R^-)^{\Sigma\cup\Delta} \cup (S^-)^{\Sigma\cup\Delta}$, *where* $\bowtie$ *is the usual natural join among ordinary relations.* □

It is instructive to observe that $(R \mathbin{\dot\bowtie} S)^-$ contains all extensions of tuples in R^- and S^-, because at least one of R and S is believed false for these extended tuples.

Definition 5 *Let* R *be a paraconsistent relation on scheme* Σ, *and* Δ *be any scheme. Then, the* projection *of* R *onto* Δ, *denoted* $\dot\pi_\Delta(R)$, *is a paraconsistent* *relation on Δ, given by $\dot{\pi}_\Delta(R)^+ = \pi_\Delta((R^+)^{\Sigma\cup\Delta})$, and $\dot{\pi}_\Delta(R)^- = \{t \in \tau(\Delta) \mid t^{\Sigma\cup\Delta} \subseteq (R^-)^{\Sigma\cup\Delta}\}$, where π_Δ is the usual projection over Δ of ordinary relations.* $\Box$

It should be noted that, contrary to usual practice, the above definition of projection is not just for subschemes. However, if $\Delta \subseteq \Sigma$, then it coincides with the intuitive projection operation. In this case, $\dot{\pi}_\Delta(R)^-$ consists of those tuples in $\tau(\Delta)$, all of whose extensions are in R^-.

Definition 6 *Let R be a paraconsistent relation on scheme Σ, and let F be any logic formula involving attribute names in Σ, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the* selection *of R by F, denoted $\dot{\sigma}_F(R)$, is a paraconsistent relation on scheme Σ, given by $\dot{\sigma}_F(R)^+ = \sigma_F(R^+)$, and $\dot{\sigma}_F(R)^- = R^- \cup \sigma_{\neg F}(\tau(\Sigma))$, where σ_F is the usual selection of tuples satisfying F.* $\Box$

3 Disjunctive Paraconsistent Relations

In this section, we present a brief overview of disjunctive paraconsistent relations and the algebraic operations on them. For a more detailed description, refer to [18].

Definition 7 A *disjunctive paraconsistent relation, R,* over the scheme Σ consists of two components $< R^+, R^- >$ where $R^+ \subseteq 2^{\tau(\Sigma)}$ and $R^- \subseteq \tau(\Sigma)$. R^+, the *positive* component, is a set of tuple sets. Each tuple set in this component represents a disjunctive positive fact. In the case where the tuple set is a singleton, we have a definite positive fact. R^-, the *negative* component consists of tuples that we refer to as definite negative tuples. Let $\mathcal{D}(\Sigma)$ represent all disjunctive paraconsistent relations over the scheme Σ. $\Box$

Definition 8 Let R be a disjunctive paraconsistent relation over Σ. Then,
$\mathbf{norm}(R)^+ = \{w | w \in R^+ \wedge w \not\subseteq R^-\}$
$\mathbf{norm}(R)^- =$
$R^- - \{t | t \in R^- \wedge (\exists w)(w \in R^+ \wedge t \in w \wedge w \subseteq R^-)\}$
$\Box$

A disjunctive paraconsistent relation is called *normalized* if it does not contain any inconsistencies. We let $\mathcal{N}(\Sigma)$ denote the set of all normalized disjunctive paraconsistent relations over scheme Σ.

Definition 9 Let R be a normalized disjunctive paraconsistent relation. Then, $\mathbf{reduce}(R)$ is defined as follows*:*
$\mathbf{reduce}(R)^+ =$
$\{w' \mid (\exists w)(w \in R^+ \wedge w' = w - R^- \wedge \neg(\exists w_1)(w_1 \in R^+ \wedge (w_1 - R^-) \subset w'))\}$
$\mathbf{reduce}(R)^- = R^-$ $\Box$

Definition 10 Let $U \subseteq \mathcal{P}(\Sigma)$. Then,
$\mathbf{normrep}_\Sigma(U) = U - \{R | R \in U \wedge R^+ \cap R^- \neq \emptyset\}$ $\Box$

The **normrep** operator removes all inconsistent paraconsistent relations from its input.

Definition 11 Let $U \subseteq \mathcal{P}(\Sigma)$. Then,
$\mathbf{reducerep}_\Sigma(U) = \{R | R \in U \wedge \neg(\exists S)(S \in U \wedge R \neq S \wedge S^+ \subseteq R^+ \wedge S^- \subseteq R^-)\}$
$\Box$

The **reducerep** operator keeps only the "minimal" paraconsistent relations and eliminates any paraconsistent relation that is "subsumed" by others.

Definition 12 The information content of disjunctive paraconsistent relations is defined by the mapping $\mathbf{rep}_\Sigma : \mathcal{N}(\Sigma) \to \mathcal{P}(\Sigma)$. Let R be a normalized disjunctive paraconsistent relation on scheme Σ with $R^+ = \{w_1, \ldots, w_k\}$. Let
$U = \{< \{t_1, \ldots, t_k\}, R^- > \mid (\forall i)(1 \le i \le k \to t_i \in w_i)\}$. Then,
$\mathbf{rep}_\Sigma(R) = \mathbf{reducerep}_\Sigma(\mathbf{normrep}_\Sigma(U))$ $\Box$

Algebraic Operators

Definition 13 Let R and S be two normalized disjunctive paraconsistent relations on scheme Σ with
$\mathbf{reduce}(R)^+ = \{v_1, \ldots, v_n\}$ and
$\mathbf{reduce}(S)^+ = \{w_1, \ldots, w_m\}$.
Then, $R\widehat{\cup}S$ is a disjunctive paraconsistent relation over scheme Σ given by
$R\widehat{\cup}S = \mathbf{reduce}(T)$, where
$T^+ = \mathbf{reduce}(R)^+ \cup \mathbf{reduce}(S)^+$ *and*
$T^- = \mathbf{reduce}(R)^- \cap \mathbf{reduce}(S)^-$,
and $R\widehat{\cap}S$ is a disjunctive paraconsistent relation over scheme Σ given by $R\widehat{\cap}S = \mathbf{reduce}(T)$, where T is defined as follows. Let
$E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \le i \le n \to t_i \in v_i)\}$ and
$F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \le i \le m \to t_i \in w_i)\}$.
Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$ for $1 \le i \le e$ and $1 \le j \le f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}s.
Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$
$T^- = R^- \cup S^-$. $\Box$

Definition 14 Let R be a normalized disjunctive paraconsistent relation on scheme Σ, and let F be any logic formula involving attribute names in Σ, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the *selection* of R by F, denoted $\widehat{\sigma}_F(R)$, is a disjunctive paraconsistent relation on scheme Σ, given by $\widehat{\sigma}_F(R) = \mathbf{reduce}(T)$, where
$T^+ = \{w | w \in \mathbf{reduce}(R)^+ \wedge (\forall t \in w)F(t)\}$ and
$T^- = \mathbf{reduce}(R)^- \cup \sigma_{\neg F}(\tau(\Sigma))$,
where σ_F is the usual selection of tuples. $\Box$

If Σ and Δ are relation schemes such that $\Sigma \subseteq \Delta$, then for any tuple $t \in \tau(\Sigma)$, we let t^{Δ} denote the set $\{t' \in \tau(\Delta) \mid t'(A) = t(A), \text{for all } A \in \Sigma\}$ of all extensions of t. We extend this notion for any $T \subseteq \tau(\Sigma)$ by defining $T^{\Delta} = \cup_{t \in T}\, t^{\Delta}$.

Definition 15 Let R be a normalized disjunctive paraconsistent relation on scheme Σ, and $\Delta \subseteq \Sigma$. Then, the *projection* of R onto Δ, denoted $\widehat{\pi}_{\Delta}(R)$, is a disjunctive paraconsistent relation on scheme Δ, given by $\widehat{\pi}_{\Delta}(R) = \mathbf{reduce}(T)$, where
$T^+ = \{\pi_{\Delta}(w) | w \in \mathbf{reduce}(R)^+\}$ and
$T^- = \{t \in \tau(\Delta) | t^{\Sigma \cup \Delta} \subseteq (\mathbf{reduce}(R)^-)^{\Sigma \cup \Delta}\}$,
where π_{Δ} is the usual projection over Δ of tuples. □

Definition 16 Let R and S be normalized disjunctive paraconsistent relations on schemes Σ and Δ, respectively with $\mathbf{reduce}(R)^+ = \{v_1, \ldots, v_n\}$ and $\mathbf{reduce}(S)^+ = \{w_1, \ldots, w_m\}$. Then, the *natural join* of R and S, denoted $R \,\widehat{\bowtie}\, S$, is a disjunctive paraconsistent relation on scheme $\Sigma \cup \Delta$, given by $R \,\widehat{\bowtie}\, S = \mathbf{reduce}(T)$, where T is defined as follows. Let
$E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$ and
$F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$.
Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \bowtie F_j$ for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}s. Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_i \wedge \cdots \wedge t_g \in A_g \wedge$
$w = \{t_1, \ldots, t_g\})\}$, and
$T^- = (\mathbf{reduce}(R)^-)^{\Sigma \cup \Delta} \cup (\mathbf{reduce}(S)^-)^{\Sigma \cup \Delta}$.

□

4 Generalized Disjunctive Paraconsistent Relations

In this section, we present the main structure underling our model, the *generalized disjunctive paraconsistent relations*. We identify several types of redundancies and inconsistencies that may appear and provide operators to remove them. Finally, we present the information content of generalized paraconsistent relations.

Definition 17 A *generalized disjunctive paraconsistent relation, R,* over the scheme Σ consists of two components $\langle R^+, R^- \rangle$ where $R^+ \subseteq 2^{\tau(\Sigma)}$ and $R^- \subseteq 2^{\tau(\Sigma)}$. R^+, the *positive* component, is a set of tuple sets. Each tuple set in this component represents a disjunctive positive fact. In the case where the tuple set is a singleton, we have a definite positive fact. R^-, the *negative* component consists of a set of tuple sets. Each tuple set in this component represents a disjunctive negative fact. In the case where the tuple set is a singleton, we have a definite negated fact. Let $\mathcal{GD}(\Sigma)$ represent all generalized disjunctive paraconsistent relations over the scheme Σ. □

Example 1 *Consider the following generalized disjunctive paraconsistent relation:*

$$\begin{aligned} supply^+ = \{&\{< s1, p1 >\}, \{< s2, p1 >, < s2, p2 >\}, \\ &\{< s3, p3 >, < s3, p4 >\}\} \\ supply^- = \{&\{< s1, p2 >\}, \{< s1, p3 >\}, \\ &\{< s2, p3 >, < s2, p4 >\}\}. \end{aligned}$$

The positive component *corresponds to the statement* $s1$ *supplies* $p1$*,* $s2$ *supplies* $p1$ *or* $p2$*, and* $s3$ *supplies* $p3$ *or* $p4$ *and the* negative component *corresponds to* $s1$ *does not supply* $p2$ *and* $s1$ *does not supply* $p3$ *and* $s2$ *does not supply* $p3$ *or* $s2$ *does not supply* $p4$*. It should be noted that the status of tuples that do not appear anywhere in the generalized disjunctive paraconsistent relation, such as* $(s3, p2)$*, is unknown.* □

Inconsistencies

Inconsistency can be present in a generalized disjunctive paraconsistent relation in two situations. On the one hand inconsistency is present if each of the tuples of a tuple set of the positive component are also present as singletons in negative component. In such a case, the positive tuple set states that at least one of the tuples in the tuple set must be in the relation whereas the negative component states that all the tuples in the tuple set must not be in the relation. We deal with this inconsistency by removing both the positive tuple set and all its corresponding singleton tuple sets from the negative component. On the other hand inconsistency is present if all the tuples of a tuple set of the negative component are also present as singletons in the positive component. In such a case, the tuple set states that at least one of the tuples in the tuple set must not be in the relation whereas the positive component states that all the tuples in the tuple set must be in the relation. We deal with this inconsistency by removing both the negative tuple set and all its corresponding singleton tuple sets from the positive component.

The **g_norm** operator defined below removes both kinds of inconsistencies from a generalized disjunctive paraconsistent relation.

Definition 18 Let R be a generalized disjunctive paraconsistent relation over Σ. Then,
$\mathbf{g\_norm}(R)^+ = R^+ -$
$\{w | w \in R^+ \wedge (\forall t)(t \in w \rightarrow \{t\} \in R^-)\} -$
$\{\{t\} | (\exists u)(u \in R^- \wedge (\forall s)(s \in u \rightarrow \{s\} \in R^+) \wedge t \in u)\}$
$\mathbf{g\_norm}(R)^- = R^- -$
$\{w | w \in R^- \wedge (\forall t)(t \in w \rightarrow \{t\} \in R^+)\} -$
$\{\{t\} | (\exists u)(u \in R^+ \wedge (\forall s)(s \in u \rightarrow \{s\} \in R^-) \wedge t \in u)\}$

□

A generalized disjunctive paraconsistent relation is called *normalized* if it does not contain any inconsistencies. We let $\mathcal{GN}(\Sigma)$ denote the set of all normalized generalized disjunctive paraconsistent relations over scheme Σ.

Redundancies

We now identify the following four types of redundancies in a normalized generalized disjunctive paraconsistent relation R:

1. $w_1 \in R^+$, $w_2 \in R^+$, and $w_1 \subset w_2$. In this case, w_1 subsumes w_2. To eliminate this redundancy, we delete w_2 from R^+.

2. $u_1 \in R^-$, $u_2 \in R^-$, and $u_1 \subset u_2$. In this case, u_1 subsumes u_2. To eliminate this redundancy, we delete u_2 from R^-.

3. $u = \{t_1, \cdots, t_k\} \in R^-$ and $\{t_i\} \in R^+$, $1 \leq i \leq n < k$. This redundancy is eliminated by deleting the tuple set u from R^- and adding the tuple set $u - \{t_1, ..., t_n\}$ to R^-. Since we are dealing with normalized generalized disjunctive paraconsistent relations, $u - \{t_1, ..., t_n\}$ cannot be empty.

4. $w = \{t_1, ..., t_k\} \in R^+$ and $\{t_i\} \in R^-$, $1 \leq i \leq n < k$. This redundancy is eliminated by deleting the tuple set w from R^+ and adding the tuple set $w - \{t_1, ..., t_n\}$ to R^+. Since we are dealing with normalized generalized disjunctive paraconsistent relations, $w - \{t_1, ..., t_n\}$ cannot be empty.

We now introduce an operator called **g_reduce** which removes the redundancies identified above.

Definition 19 Let R be a normalized generalized disjunctive paraconsistent relation and let
$U = \{t | \{t\} \in R^-\}$ *and* $W = \{t | \{t\} \in R^+\}$. Then,

$$\textbf{g\_reduce}(R)^+ = \{w' | (\exists w)(w \in R^+ \land w' = w - U \land \neg(\exists w_1)(w_1 \in R^+ \land (w_1 - U) \subset w'))\}$$

$$\textbf{g\_reduce}(R)^- = \{u' | (\exists u)(u \in R^- \land u' = u - W \land \neg(\exists u_1)(u_1 \in R^- \land u_1 - W) \subset u'))\}$$

□

Example 2 *Consider the following generalized disjunctive paraconsistent relation:*

$$R^+ = \{\{< a >\}, \{< b >, < c >\}, \{< c >, < d >\}, \{< a >, < e >\}, \{< f >, < g >\}\}$$

and $R^- = \{\{< b >\}, \{< c >, < e >\}, \{< i >\}, \{< d >, < e >, < f >\}\}$.

The disjunctive tuple $\{< a >, < e >\}$ *is subsumed by* $\{< a >\}$ *and hence removed. In the disjunctive tuple set* $\{< b >, < c >\}$, $< b >$ *is redundant due to the presence of the negative singleton tuple set* $\{< b >\}$ *resulting in the positive tuple* $\{< c >\}$ *which in turn subsumes* $\{< c >, < d >\}$ *and makes* $\{< c >, < e >\}$ *redundant and resulting in* $\{< e >\}$ *which subsumes the* $\{< d >, < e >, < f >\}$. *The reduced generalized disjunctive paraconsistent relation is:*

$$\textbf{g\_reduce}(R)^+ = \{\{< a >\}, \{< c >\}, \{< f >, < g >\}\} \textit{ and}$$
$$\textbf{g\_reduce}(R)^- = \{\{< b >\}, \{< e >\}, \{< i >\}\}$$ □

Information Content

The information content of a generalized disjunctive paraconsistent relation can be defined to be a collection of disjunctive paraconsistent relations. The different possible disjunctive paraconsistent relations are constructed by selecting one of the several tuples within a tuple set for each tuple set in the negative component. In doing so, we may end up with non-minimal disjunctive paraconsistent relations or even with inconsistent disjunctive paraconsistent relations. These would have to be removed in order to obtain the exact information content of generalized disjunctive paraconsistent relations. The formal definitions follow:

Definition 20 Let $U \subseteq \mathcal{D}(\Sigma)$. Then,

$$\textbf{g\_normrep}_\Sigma(U) = \{R | R \in U \land \neg(\exists w)(w \in R^+ \land w \subseteq R^-)\}$$ □

The **g_normrep** operator removes all inconsistent disjunctive paraconsistent relations from its input.

Definition 21 Let $U \subseteq \mathcal{D}(\Sigma)$. Then,

$$\textbf{g\_reducerep}_\Sigma(U) = \{R | R \in U \land \neg(\exists S)(S \in U \land R \neq S \land S^+ \subseteq R^+ \land S^- \subseteq R^-)\}$$

□

The **g_reducerep** operator keeps only the "minimal" disjunctive paraconsistent relations and eliminates any disjunctive paraconsistent relation that is "subsumed" by others.

Definition 22 The information content of generalized disjunctive paraconsistent relations is defined by the mapping $\textbf{g\_rep}_\Sigma : \mathcal{GN}(\Sigma) \to \mathcal{D}(\Sigma)$. Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ *with* $R^- = \{u_1, \ldots, u_m\}$.
Let $U = \{< R^+, \{t_1, \ldots, t_m\} > | (\forall i)(1 \leq i \leq m \to t_i \in u_i)\}$. Then,

$$\textbf{g\_rep}_\Sigma(R) = \textbf{g\_reducerep}_\Sigma(\textbf{g\_normrep}_\Sigma(U))$$ □

Note that the information content is defined only for normalized generalized disjunctive paraconsistent relations.

The following important theorem states that information is neither lost nor gained by removing the redundancies in a generalized disjunctive paraconsistent relations.

Theorem 1 *Let* R *be a generalized disjunctive paraconsistent relation on scheme* Σ. *Then,*

$$\textbf{g\_rep}_\Sigma(\textbf{g\_reduce}(R)) = \textbf{g\_rep}_\Sigma(R)$$ □

5 Generalized Relational Algebra

$\square$

In this section, we first develop the notion of *precise generalizations* of algebraic operators. This is an important property that must be satisfied by any new operator defined for generalized disjunctive paraconsistent relations. Then, we present several algebraic operators on generalized disjunctive paraconsistent relations that are precise generalizations of their counterparts on disjunctive paraconsistent relations.

Precise Generalization of Operations

We now construct a framework for operators on both kinds of relations and introduce the notion of the precise generalization relationship among their operators.

An n-ary *operator on disjunctive paraconsistent relations with signature* $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a function $\Theta \ : \ \mathcal{D}(\Sigma_1) \times \cdots \times \mathcal{D}(\Sigma_n) \rightarrow \mathcal{D}(\Sigma_{n+1})$, where $\Sigma_1, \ldots, \Sigma_{n+1}$ are any schemes. Similarly, an n-ary *operator on generalized disjunctive paraconsistent relations with signature* $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a function: $\Psi : \mathcal{GD}(\Sigma_1) \times \cdots \times \mathcal{GD}(\Sigma_n) \rightarrow \mathcal{GD}(\Sigma_{n+1})$.

We now need to extend operators on disjunctive paraconsistent relations to sets of disjunctive paraconsistent relations. For any operator $\Theta \ : \ \mathcal{D}(\Sigma_1) \times \cdots \times \mathcal{D}(\Sigma_n) \rightarrow \mathcal{D}(\Sigma_{n+1})$ on disjunctive paraconsistent relations, we let $\mathcal{S}(\Theta) \ : \ 2^{\mathcal{D}(\Sigma_1)} \times \cdots \times 2^{\mathcal{D}(\Sigma_n)} \rightarrow 2^{\mathcal{D}(\Sigma_{n+1})}$ be a map on sets of disjunctive paraconsistent relations defined as follows. For any sets $M_1, \ldots, M_n$ of disjunctive paraconsistent relations on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively,

$$\mathcal{S}(\Theta)(M_1, \ldots, M_n) = \{\Theta(R_1, \ldots, R_n) | R_i \in M_i, \text{for all } i, 1 \leq i \leq n\}.$$

In other words, $\mathcal{S}(\Theta)(M_1, \ldots, M_n)$ is the set of Θ-images of all tuples in the Cartesian product $M_1 \times \cdots \times M_n$. We are now ready to lead up to the notion of precise operator generalization.

Definition 23 An operator Ψ on generalized disjunctive paraconsistent relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is *consistency preserving* if for any normalized generalized disjunctive relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively, $\Psi(R_1, \ldots, R_n)$ is also normalized. $\square$

Definition 24 A consistency preserving operator Ψ on generalized disjunctive paraconsistent relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a *precise generalization* of an operator Θ on disjunctive paraconsistent relations with the same signature, if for any normalized generalized disjunctive paraconsistent relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, we have $\mathbf{g\_rep}_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = \mathcal{S}(\Theta)(\mathbf{g\_rep}_{\Sigma_1}(R_1), \ldots, \mathbf{g\_rep}_{\Sigma_n}(R_n))$.

We now present precise generalizations for the usual relation operators, such as union, join, projection. To reflect generalization, a line is placed over an ordinary operator. For example, $\bowtie$ denotes the natural join among ordinary relations, $\dot{\bowtie}$ denotes natural join on paraconsistent relations, $\hat{\bowtie}$ denotes natural join on disjunctive paraconsistent relations and $\overline{\bowtie}$ denotes natural join on generalized disjunctive paraconsistent relations.

Union and Intersection

Definition 25 Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme Σ with
$\mathbf{g\_reduce}(R)^+ = \{u_1, \ldots, u_n\}$,
$\mathbf{g\_reduce}(R)^- = \{v_1, \ldots, v_k\}$,
$\mathbf{g\_reduce}(S)^+ = \{w_1, \ldots, w_m\}$, and
$\mathbf{g\_reduce}(S)^- = \{x_1, \ldots, x_j\}$.
Then, $R \overline{\cup} S$ *is a generalized disjunctive paraconsistent relation over scheme* Σ *given by* $R \overline{\cup} S = \mathbf{g\_reduce}(T)$, where T is defined as follows.
Let $E = \{\{t_1, \ldots, t_k\} | (\forall i)(1 \leq i \leq k \rightarrow t_i \in v_i)\}$ and $F = \{\{t_1, \ldots, t_j\} | (\forall i)(1 \leq i \leq j \rightarrow t_i \in x_i)\}$.
Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$, for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}s. Then,
$T^+ = \mathbf{g\_reduce}(R)^+ \cup \mathbf{g\_reduce}(S)^+$
$T^- = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$.
and $R \overline{\cap} S$ *is a generalized disjunctive paraconsistent relation over scheme* Σ *given by* $R \overline{\cap} S = \mathbf{g\_reduce}(T)$, where T is defined as follows.
Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in u_i)\}$ and $F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$.
Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$, for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij} s. Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$.
$T^- = \mathbf{g\_reduce}(R)^- \cup \mathbf{g\_reduce}(S)^-$. $\square$

The following theorem establishes the *precise generalization* property for union and intersection:

Theorem 2 *Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme Σ. Then,*

1. $\mathbf{g\_rep}_\Sigma(R \overline{\cup} S) = \mathbf{g\_rep}_\Sigma(R) \mathcal{S}(\hat{\cup}) \mathbf{g\_rep}_\Sigma(S)$.
2. $\mathbf{g\_rep}_\Sigma(R \overline{\cap} S) = \mathbf{g\_rep}_\Sigma(R) \mathcal{S}(\hat{\cup}) \mathbf{g\_rep}_\Sigma(S)$. $\square$

Complement

Definition 26 Let R be normalized generalized disjunctive paraconsistent relation on scheme Σ.

Then, $\overline{-}R$ is a generalized disjunctive paraconsistent relation over scheme Σ given by
$(\overline{-}R)^+ = \mathbf{g\_reduce}(R)^-$ and $(\overline{-}R)^- = \mathbf{g\_reduce}(R)^+$.

$\square$

Selection

Definition 27 Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ, and let F be any logic formula involving attribute names in Σ, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the *selection of R by F, denoted* $\overline{\sigma}_F(R)$, is a generalized disjunctive paraconsistent relation on scheme Σ, given by $\overline{\sigma}_F(R) = \mathbf{g\_reduce}(T)$, where
$T^+ = \{w | w \in \mathbf{g\_reduce}(R)^+ \wedge (\forall t \in w) F(t)\}$ and
$T^- = R^- \cup \sigma_{\neg F}(\tau(\Sigma))$, where σ_F is the usual selection of tuples.

$\square$

A disjunctive tuple set is either selected as a whole or not at all. All the tuples within the tuple set must satisfy the selection criteria for the tuple set to be selected.

Project

Definition 28 Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ with $\mathbf{g\_reduce}(R)^- = \{v_1, \ldots, v_n\}$., and $\Delta \subseteq \Sigma$. Then, the *projection of R onto* Δ, *denoted* $\overline{\pi}_\Delta(R)$, is a generalized disjunctive paraconsistent relation on scheme Δ, given by $\overline{\pi}_\Delta(R) = \mathbf{g\_reduce}(T)$, where T is defined as follows.
Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \le i \le n \rightarrow t_i \in v_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and let
$A_i = \{t \in \pi(\Delta) | t^{\Sigma \cup \Delta} \subseteq (E_i)^{\Sigma \cup \Delta}\}$. *Then,*
$T^+ = \{\pi_\Delta(w) | w \in \mathbf{g\_reduce}(R)^+\}$
$T^- = \{w | (\exists t_1) \ldots (\exists t_e)(t_1 \in A_i \wedge \ldots \wedge t_e \in A_e \wedge w = \{t_1, \ldots, t_e\})\}$, where π_Δ is the usual projection over Δ of tuples.

$\square$

The positive component of the projections consists of the projection of each of the tuple sets onto Δ and $\overline{\pi}_\Delta(R)^-$ consists of those tuple sets in $\tau(\Delta)$, all of whose extensions are in R^-.

Natural Join

Definition 29 Let R and S be normalized generalized disjunctive paraconsistent relations on schemes Σ and Δ, respectively with
$\mathbf{g\_reduce}(R)^+ = \{u_1, \ldots, u_n\}$,
$\mathbf{g\_reduce}(R)^- = \{v_1, \ldots, v_k\}$,
$\mathbf{g\_reduce}(S)^+ = \{w_1, \ldots, w_m\}$, and
$\mathbf{g\_reduce}(S)^- = \{x_1, \ldots, x_j\}$.
Then, the *natural join of R* and S, denoted $R \overline{\bowtie} S$, is a generalized disjunctive paraconsistent relation on scheme $\Sigma \cup \Delta$, given by $R \overline{\bowtie} S = \mathbf{g\_reduce}(T)$, where T is defined as follows. Let
$E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \le i \le n \rightarrow t_i \in u_i)\}$ and
$F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \le i \le m \rightarrow t_i \in w_i)\}$.
Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \bowtie F_j$ for $1 \le i \le e$ and $1 \le j \le f$. Let $A_1, \ldots, A_g$ be the distinct $A_{ij}s$. Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$. Let
$G = \{\{t_1, \ldots, t_k\} | (\forall i)(1 \le i \le k \rightarrow t_i \in v_i)\}$ and
$H = \{\{t_1, \ldots, t_j\} | (\forall i)(1 \le i \le j \rightarrow t_i \in x_i)\}$.
Let the elements of G be $G_1, \ldots, G_g$ and those of H be $H_1, \ldots, H_h$ and let
$B_{ij} = (G_i)^{\Sigma \cup \Delta} \cup (H_j)^{\Sigma \cup \Delta}$ for $1 \le i \le g$ and $1 \le j \le h$.
Let $B_1, \ldots, B_f$ be the distinct $B_{ij}s$. Then,
$T^- = \{w | (\exists t_1) \cdots (\exists t_f)(t_1 \in B_1 \wedge \cdots \wedge t_f \in B_f \wedge w = \{t_1, \ldots, t_f\})\}$. $\square$

Theorem 3 *Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme* Σ_1 *and* Σ_2. *Also let F be a selection formula on scheme* Σ_1 *and* $\Delta \subseteq \Sigma_1$. *Then,*

1. $\mathbf{g\_rep}_{\Sigma_1}(\overline{\sigma}_F(R)) = \mathcal{S}(\widehat{\sigma}_F)(\mathbf{g\_rep}_{\Sigma_1}(R))$.

2. $\mathbf{g\_rep}_{\Sigma_1}(\overline{\pi}_\Delta(R)) = \mathcal{S}(\widehat{\pi}_\Delta)(\mathbf{g\_rep}_{\Sigma_1}(R))$.

3. $\mathbf{g\_rep}_{\Sigma_1 \cup \Sigma_2}(R \overline{\bowtie} S) = \mathbf{g\_rep}_{\Sigma_1}(R) \mathcal{S}(\widehat{\bowtie}) \mathbf{g\_rep}_{\Sigma_2}(S)$.

$\square$

6 Conclusions and Future Work

We have presented a framework for relational databases under which positive disjunctive as well as explicit negative disjunctive facts can be represented and manipulated. It is the generalization of disjunctive paraconsistent relation in [18]. There are at least two directions for future work. One would be to make the model more expressive by considering disjunctive positive and negative facts. Work is in progress in this direction. The extended model will be more expressive. The algebraic operators will have to be extended appropriately. The other direction for future work would be to design query processing for the model presented in this paper. The database management systems based on this model can be used to store and retrieve incomplete and inconsistent information existing in many real situations such as bioinformatics, biomedical application and Web intelligence, etc. There has been some interest in studying extended disjunctive logic programs in which the head of clauses can have one or more literals [13]. This leads to two notions of negation: *implicit* negation (corresponding to negative literals in the body) and *explicit* negation (corresponding to negative literals in the head). The model presented in this paper could provide a framework under which the semantics of extended disjunctive logic programs could be constructed in a bottom-up manner.

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