Computer Programming

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Session: Reasoning about loops

Quick Recap of Relevant Topics

- Sequential and conditional execution of statements
- Iteration/looping constructs
- Solving simple problems with iteration constructs in C++

Overview of This Lecture

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- Reasoning about loops as a design and post-design activity
  - Pre-conditions
  - Post-conditions
  - Loop invariants
  - Loop variants

Reasoning About Loops

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- Loop: Part of program with an iterative construct
  - Won't distinguish between "for …", "while …", "do … while …" loops for this discussion

- What does a loop compute ?
  - Enter the loop with some relation among variables : **pre-condition**
  - Exit the loop with some relation among variables: **post-condition**
  - Loop incrementally changes variables such that we move pre-condition to post-condition

Recall: Min/Max Example

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Given positive integers m and n, find $3^{min(m, n)}$ and $2^{max(m, n)}$

```
int minMN = 0, maxMN = 0, threeRaisedMin = 1, twoRaisedMax = 1;
int m, n, i, j;
for (i = m, j = n; ((i >= 1) || (j >= 1)); i--, j--)   // Iterate max(m, n) times
 {  if ((i >= 1) && (j >= 1)) {
      minMN++;  threeRaisedMin *= 3; // Conditionally iterate min(m,n) times
    }
    maxMN++;   twoRaisedMax *= 2;   // Executed max(m, n) times
 }
```

Recall: Min/Max Example

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Given positive integers m and n, find $3^{min(m, n)}$ and $2^{max(m, n)}$

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// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
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for (i = m, j = n; ((i >= 1) || (j >= 1)); i--, j--)   // Iterate max(m, n) times
 {  if ((i >= 1) && (j >= 1)) {
      minMN++;  threeRaisedMin *= 3; // Conditionally iterate min(m,n) times
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 }
// POSTCONDITION: minMN = min(m, n), maxMN = max(m, n)
//                    twoRaisedMax = 2^max(m,n), threeRaisedMin = 3^min(m,n)
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Reasoning About Loops

Given positive integers m and n, find $3^{min(m, n)}$ and $2^{max(m, n)}$

// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
// twoRaisedMax = 1, threeRaisedMin = 1

Loop incrementally changes variables such that starting from pre-condition, eventually post-condition holds

// POSTCONDITION: minMN = min(m, n), maxMN = max(m, n)
// twoRaisedMax = $2^{max(m,n)}$, threeRaisedMin = $3^{min(m,n)}$

Designing Loops: A Design Activity

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Given positive integers m and n, find $3^{min(m, n)}$ and $2^{max(m, n)}$

// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
// twoRaisedMax

```
for (i = m, j = n; ((
 { if ((i >= 1)
     minMN
   }
   maxMN,  twoRa
 }
```

How do we incrementally change variables to effect transformation from pre-condition to post-condition?

// POSTCONDITION: minMN = min(m, n), maxMN = max(m, n)
// twoRaisedMax = $2^{max(m,n)}$, threeRaisedMin = $3^{min(m,n)}$

Verifying Loops: A Post-Design Activity

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Given positive integers m and n, find $3^{min(m, n)}$ and $2^{max(m, n)}$

```
// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
//                          twoRaisedMax = 1
for (i = m, j = n; ((i >=
 {  if ((i >= 1) &&
      minMN++;
    }
    maxMN++;  twoRaised
 }
// POSTCONDITION: minMN = min(m, n), maxMN = max(m, n)
//                          twoRaisedMax = 2^max(m,n), threeRaisedMin = 3^min(m,n)
```

Does this loop really effect transformation from pre-condition to post-condition?

Reasoning About Loops

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- Designing and verifying loops requires similar kind of reasoning
- A loop iteratively transforms relations between variables such that
  - **Pre-condition** holds when we start iterating for first time
  - **Post-condition** holds when we exit loop
  - Pre-condition, post-condition special cases of relation that holds invariantly every time we are about to iterate: **loop-invariant**
  - **Desirable:** Integer valued "metric" (e.g. value of counter) monotonically changes towards fixed value : **loop-variant** (ensures loop termination)

Reasoning About Loops

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```
// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
//                        twoRaisedMax = 1, three
// LOOP INVARIANT:
//
// LOOP VARIANT:
for (i = m, j = n; ((i >= 1) || (j >= 1));  i--, j--)
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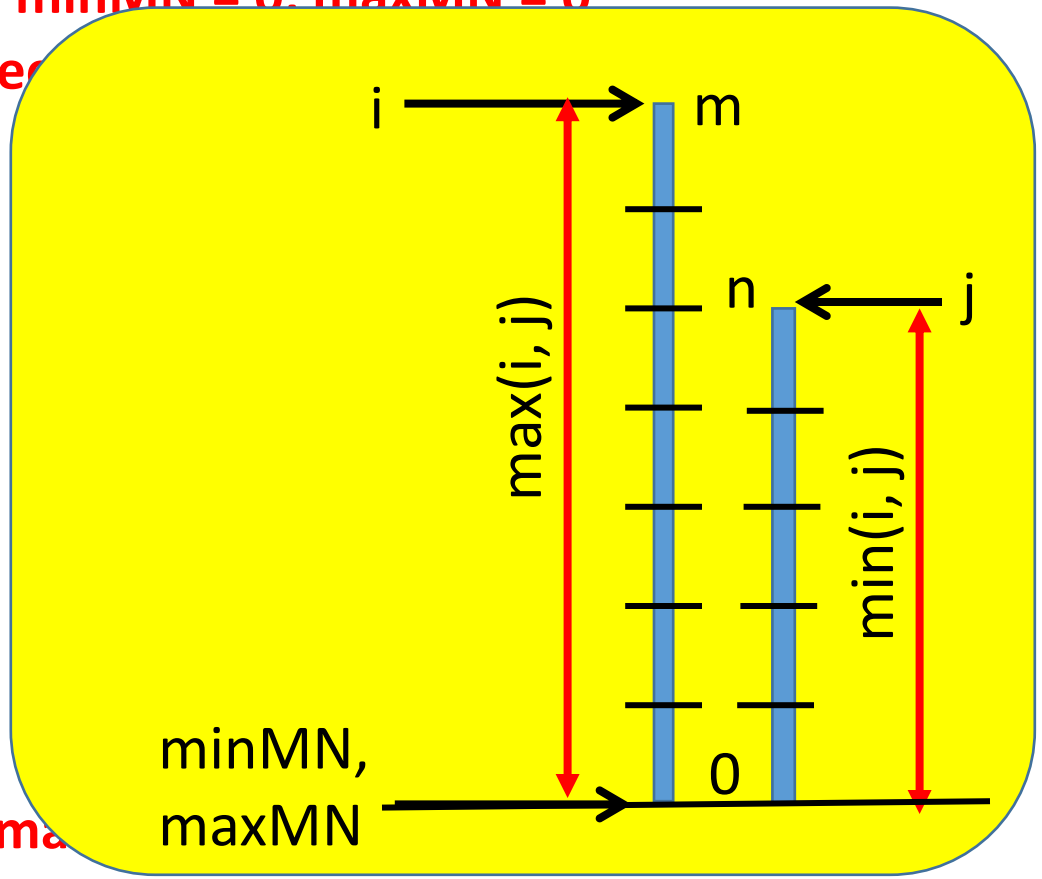

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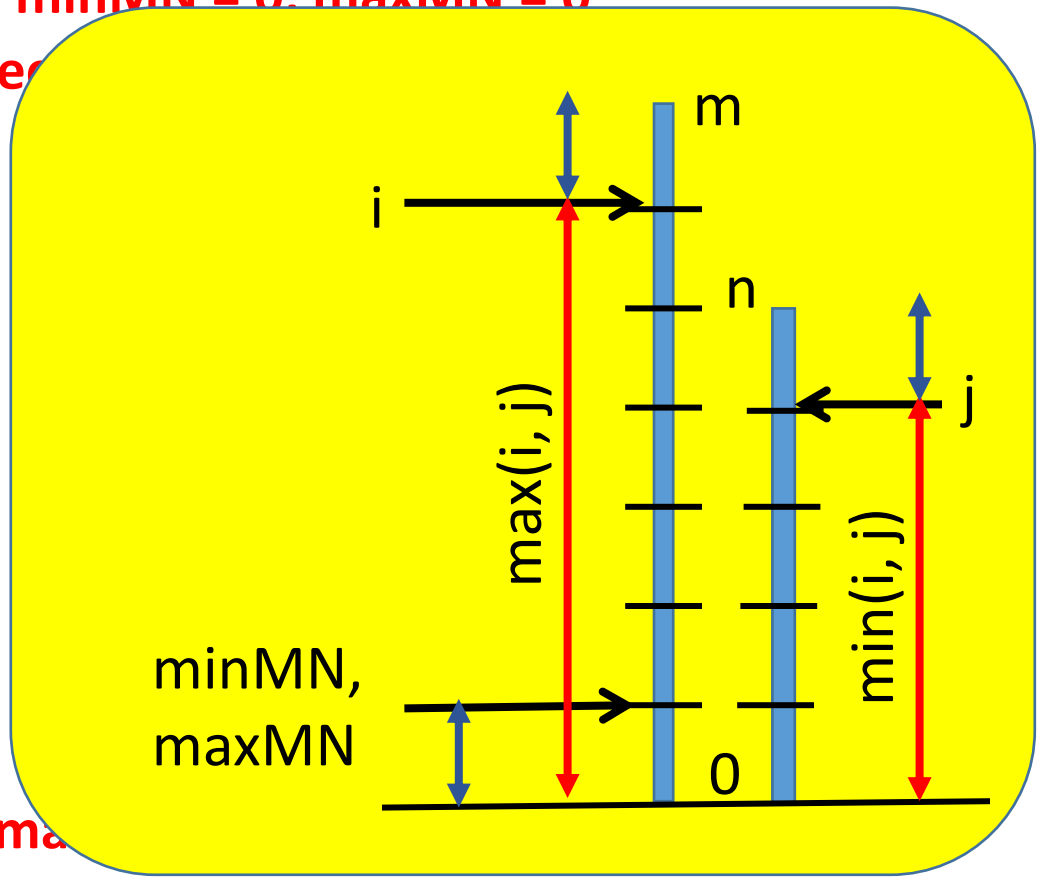

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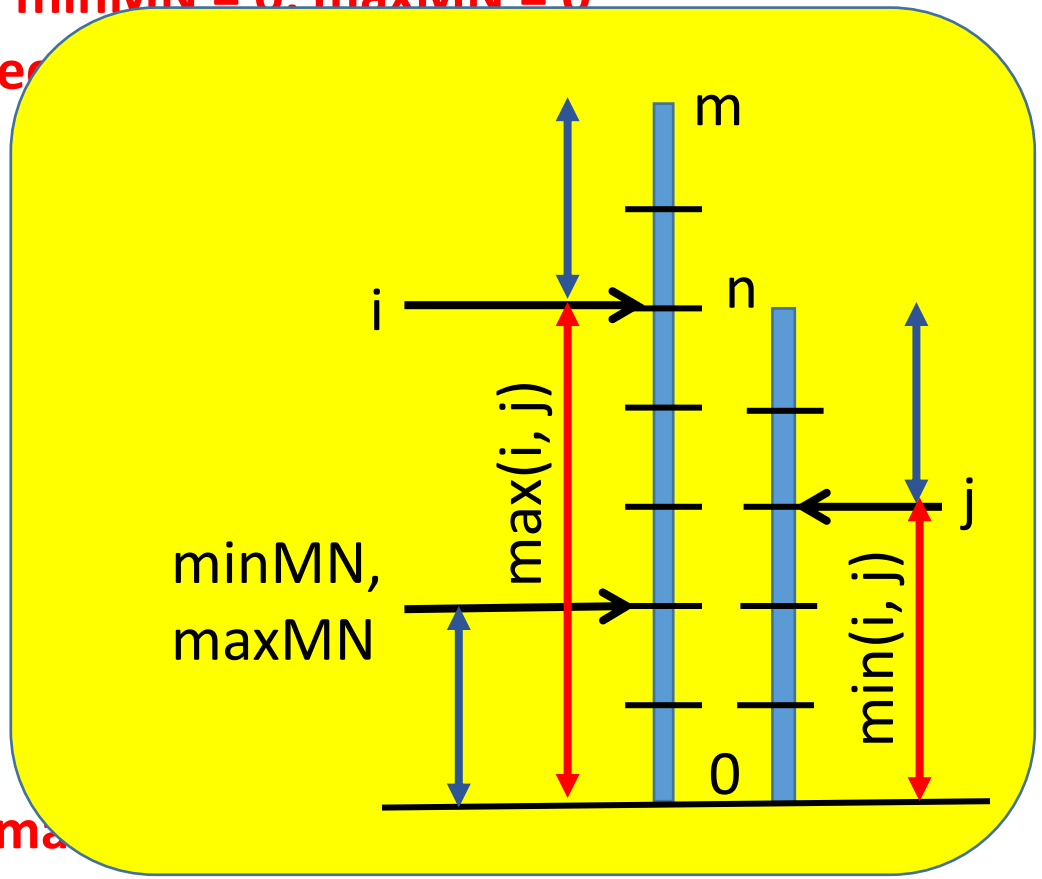

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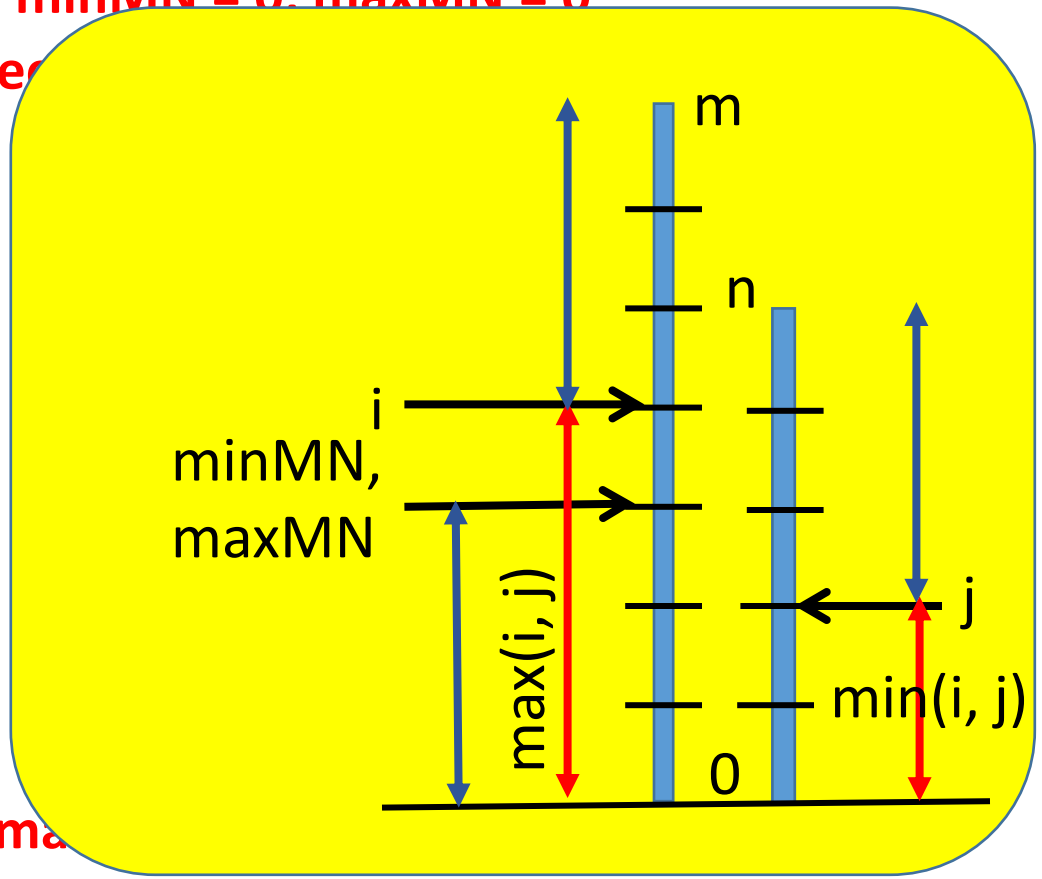

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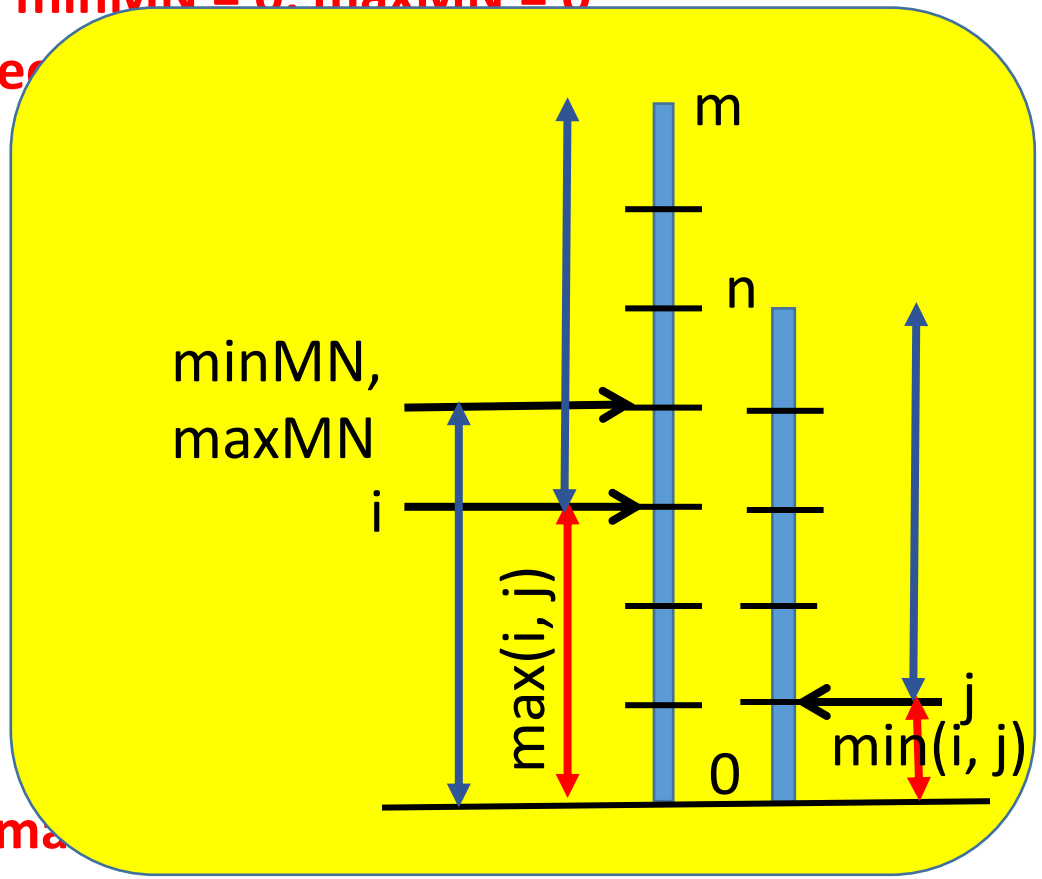

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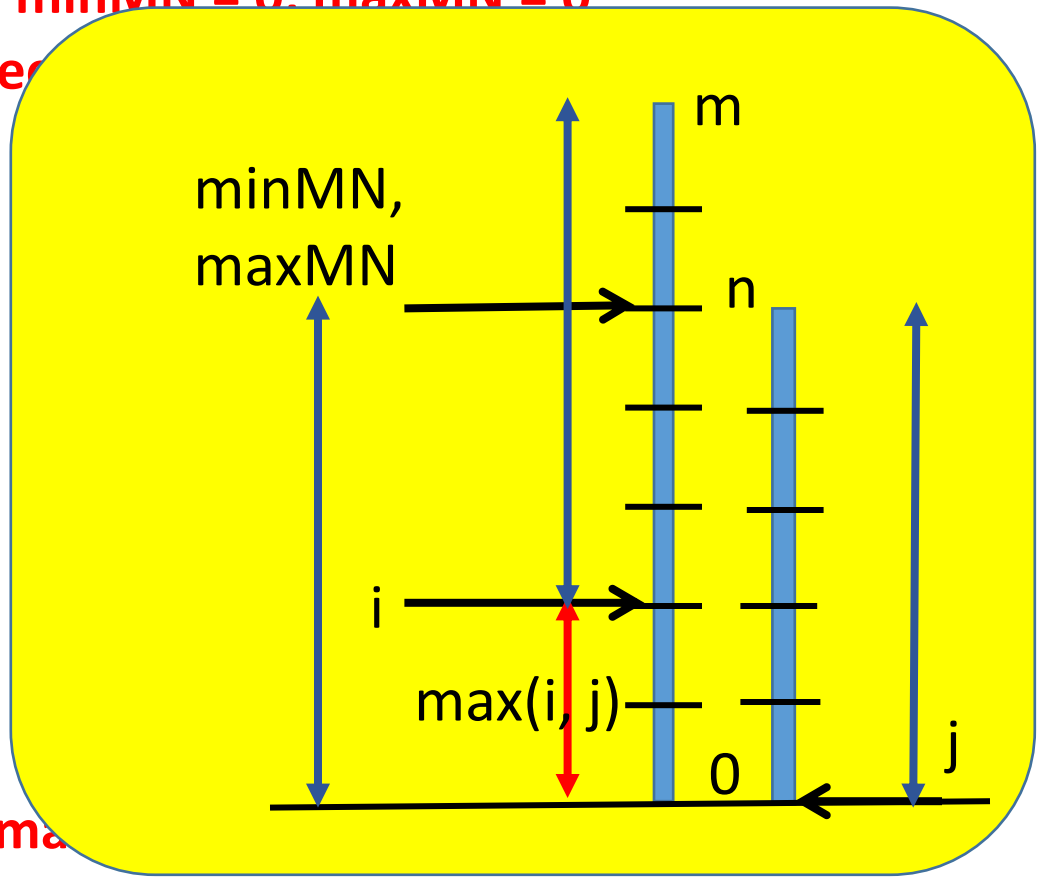

Reasoning About Loops


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Reasoning About Loops

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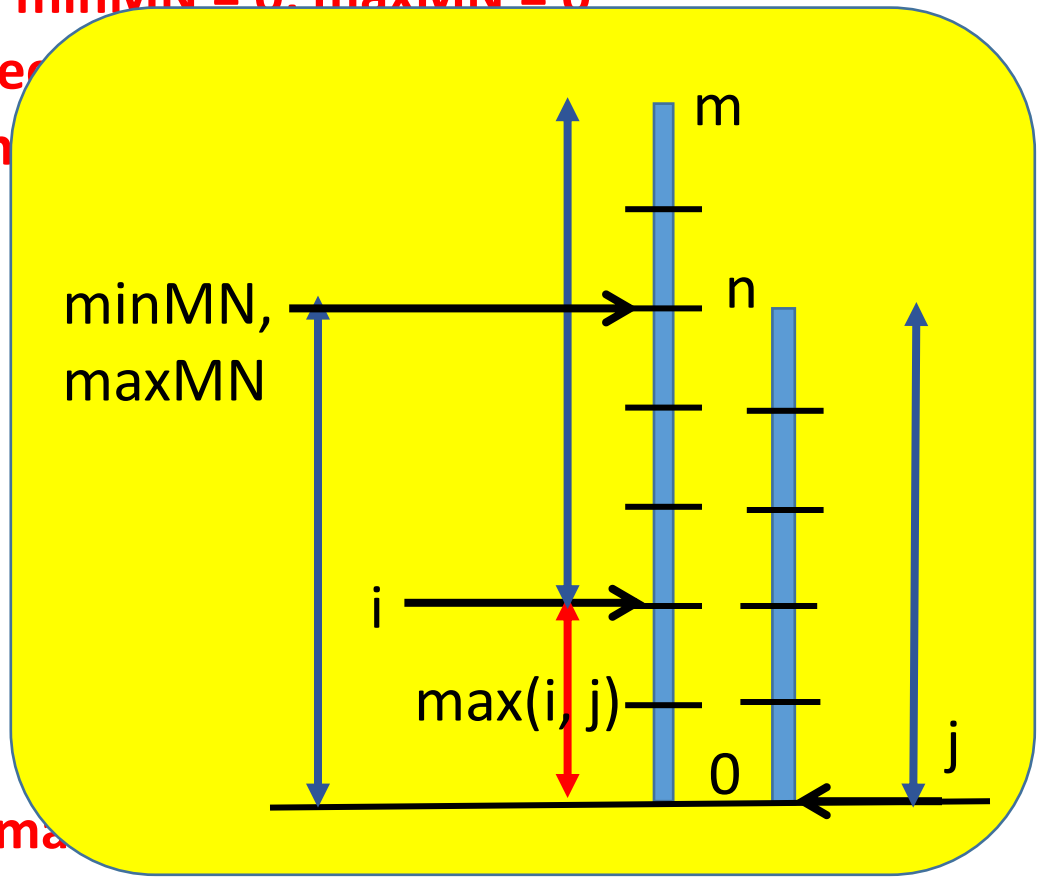

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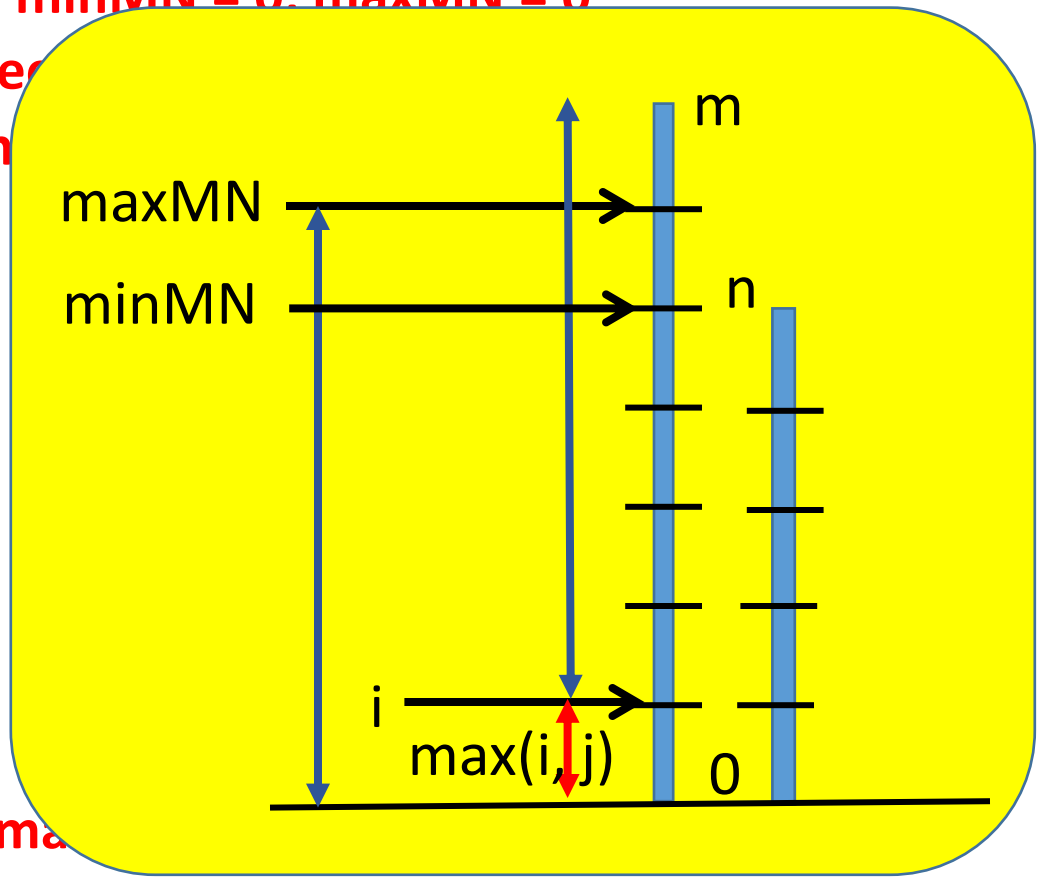

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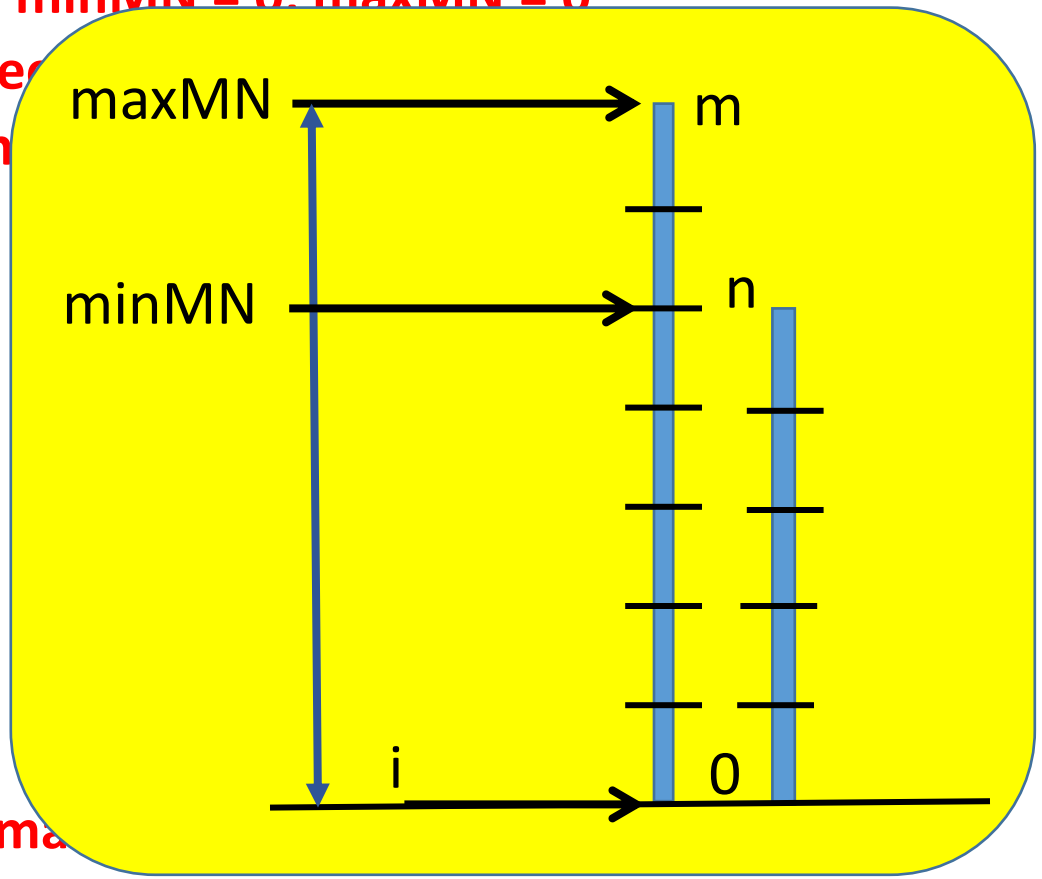

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Reasoning About Loops

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```

Taking care of "when both i, j >= 0"

Reasoning About Loops

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```

threeRaisedMin = 3^{minMN}

twoRaisedMax = 2^{maxMN}

Reasoning About Loops

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```

Non-negative integer-valued expression that monotonically decreases towards 0 in every iteration

Hence, loop terminates

Reasoning About Loops


```
// PRECONDITION: integers m >=1, n >= 1, minMN = 0, maxMN = 0
//                        twoRaisedMax = 1, threeRaisedMin = 1
// LOOP INVARIANT: max(0, min(i,j)) + minMN = min(m, n)
//                          max(i,j)  + maxMN = max(m, n)
// LOOP VARIANT: max(i, j)
```

Obtaining right loop invariant/variant after design not always easy, but crucial to reason about loops

Best practice: Write pre-conditions, post-conditions, loop invariants, loop variants as comments when programming

```
// POSTCONDITION: minMN = min(m, n), maxMN = max(m, n)
//                        twoRaisedMax = 2^max(m,n), threeRaisedMin = 3^min(m,n)
```

Summary

- Reasoning about loops in programs
- Pre-conditions, post-conditions, loop invariants and loop variants
- Importance of comments