CS206 Lecture 29

Formal Methods in Computer Science

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Plan for Lecture 29

- Kripke Structures
- Kripke Semantics

Examples of Modalities

- necessarily true
- known to be true
- believed to be true
- true in the true

i. Bondevik is prime-minister of Norway.
ii. There are 9 planets in the Solar system.
iii. The square root of 81 is 9.

ad(i)true now,will not be true in the future.
ad(ii)true now, may be true forever in the future is not necessarily true.
(it could be a different number)
ad(iii)true now, necessarily true and will be in the future.

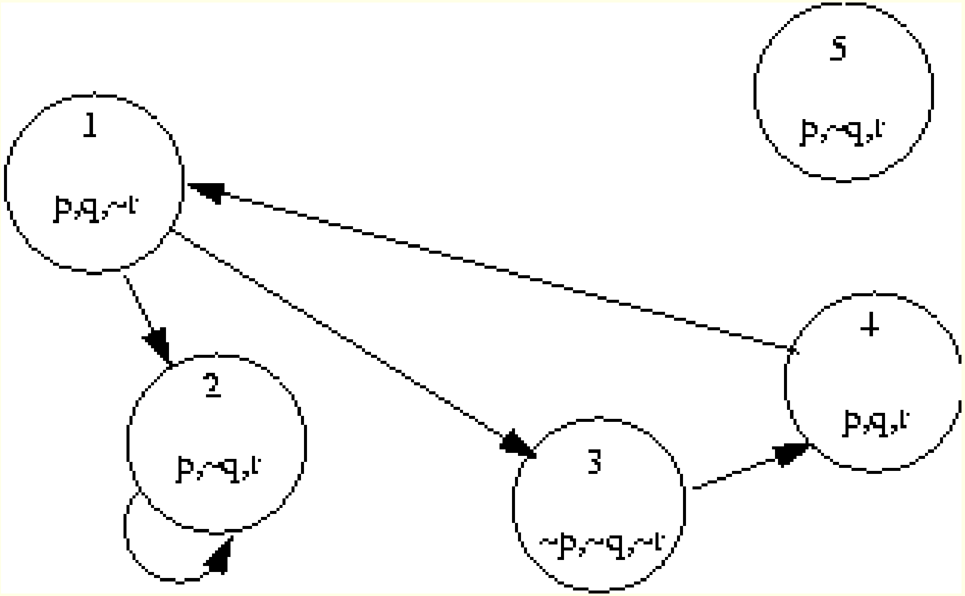
Semantics

A Kripke structure has the following components:

- A set W whose elements are called worlds.
  (Represented by circles in the diagram).
- A binary Accessibility Relation R(a,b) connecting the set of worlds.
  R(a,b) means that 'world *b* is accessible from *a*'.
- A function $L : W \longrightarrow R$ called Labelling Function which contains all the propositions that are true in a given world.

Example of Kripke Structure

In the following figure:



In the above structure

- Set of Worlds **W** : (W_1 ,W_2 , W_3 ,W_4 ,W_5)
- Accessibilty relation **R** : { (W_1 , W_2) , (W_1 ,W_3) , (W_2 , W_2) , (W_3 , W_4) }
- Labelling function **L**: $W_1 \rightarrow (p, q, \neg r)$, $W_2 \rightarrow (p, \neg q, r)$ $W_3 \rightarrow (\neg p, \neg q, \neg r)$, $W_4 \rightarrow (p, q, r)$, $W_5 \rightarrow (p, \neg q, \neg r)$

Well Formed Formulae (Wff)

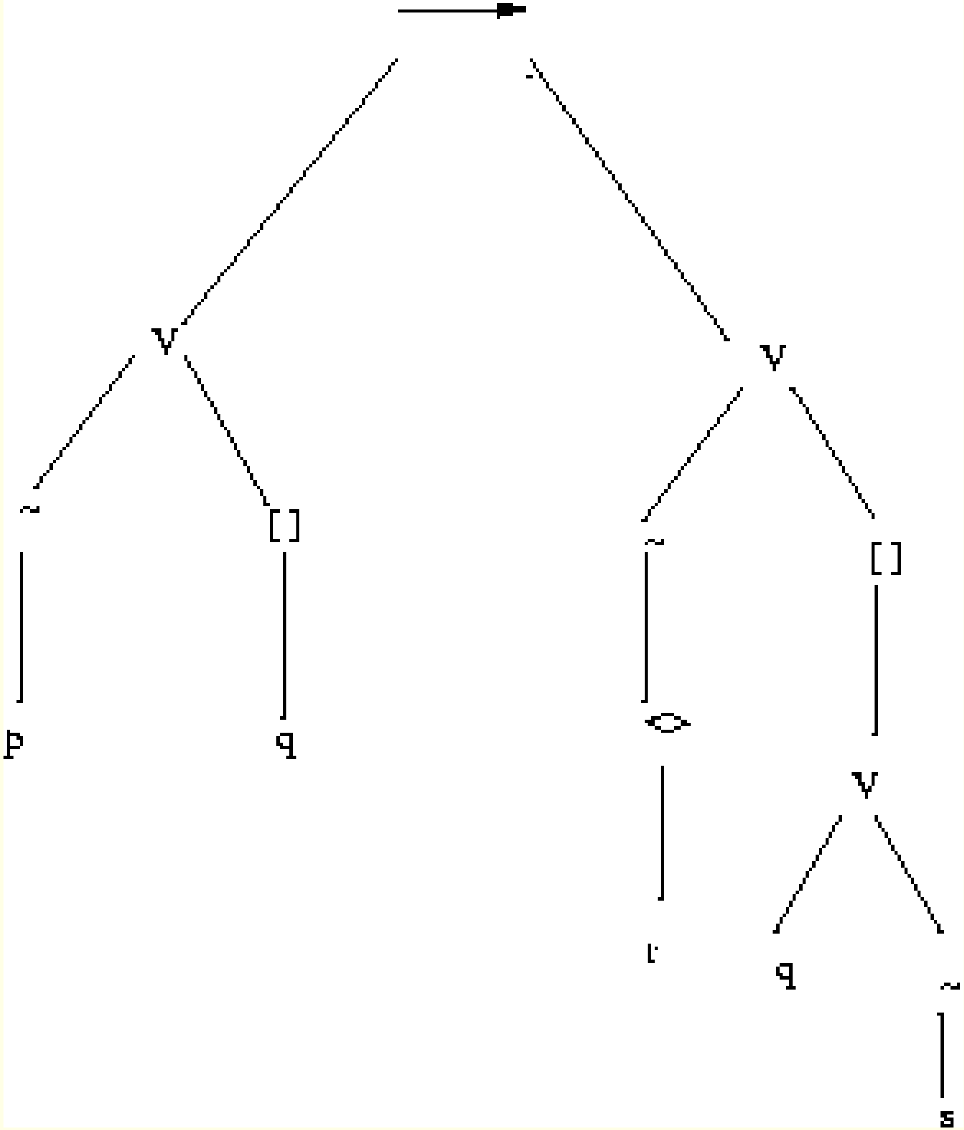
Base case: A propositional symbol f is a well formed formula.

Induction: If f1 and f2 are well formed formulae so are:

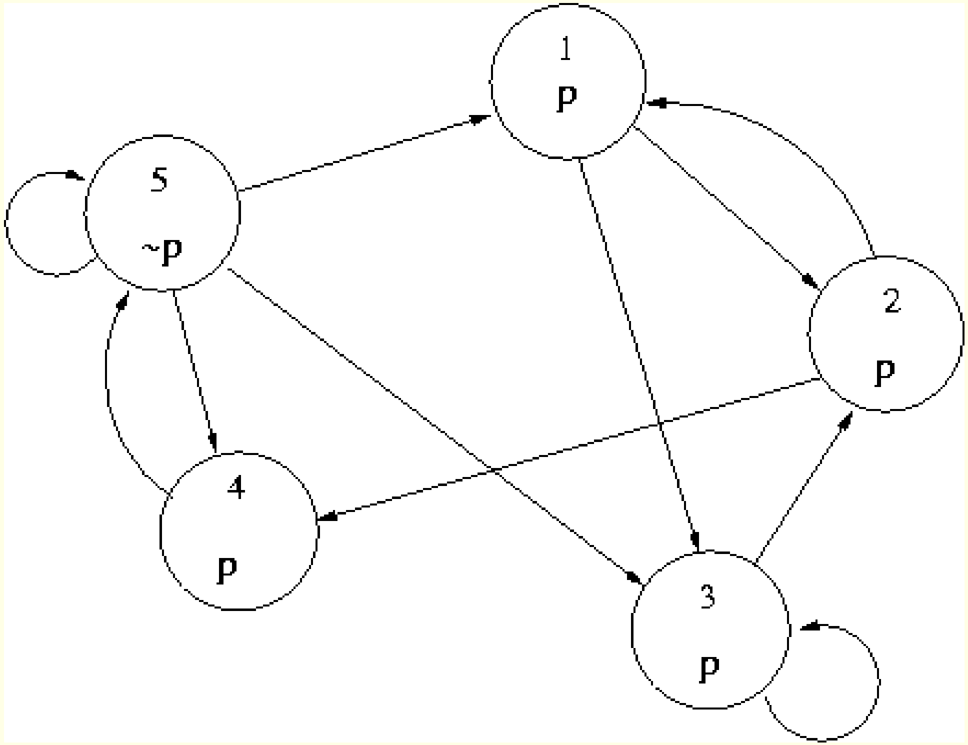
- $(f_1) \wedge (f_2)$
- $(f_1) \vee (f_2)$
- $(f_1) \longrightarrow (f_2)$
- $\neg f_1$
- $[] f_1$
- $\Diamond f_1$

Closure condition: Nothing else is a well formed formula.

The Wff $((\neg p \vee []q) \rightarrow ((\neg \Diamond r) \vee ([](q \vee \neg s))))$ can be represented as



Example of Semantics



In the above Kripke structure:

Worlds	$\Diamond P$	$[]P$	$\Diamond\Diamond P$	$\Diamond[]$P	$[]\Diamond$P	$[][]P$
W_1	T	T	T	T	T	T
W_2	T	T	T	T	T	T
W_3	T	T	T	T	T	T
W_4	F	F	T	F	F	F
W_5	T	F	T	T	F	F

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Satisfaction Relations for Kripke Structures

- $k \models_w p$ iff $p \in L(w)$
- $k \models_w \neg F$ iff $k \not\models_w F$
- $k \models_w (F \wedge G)$ iff $k \models_w F$ and $k \models_w G$
- $k \models_w (F \vee G)$ iff $k \models_w G$ or $k \models_w F$
- $k \models_w F \rightarrow G$ iff $k \models_w G$,whenever we have $k \models_w F$
- $k \models_w F \iff G$ iff $(k \models_w F$ iff $k \models_w G)$
- $k \models_w []F$ iff, for each $y \in W$ with $R(x, y)$ we have $y \models_w F$
- $k \models_w \Diamond F$ iff, there is $y \in W$ such that $R(x, y)$ and $y \models_w F$
- $k \models p$ iff p is true in all worlds of k

Kripke Structure K

All Kripke structures satisfy the property:

$\models [](F \rightarrow G) \rightarrow ([]F \rightarrow []G)$

Proof:

Suppose the above is not true.

$\Rightarrow \exists$ a world W_i where $[](F \rightarrow G)$ is true,
but $([]F \rightarrow []G)$ is not true

$\Rightarrow []F$ is true in W_i, $[]G$ is false in W_i

$\Rightarrow \exists$ a W_j such that $(W_i, W_j) \in$ R and F is true in W_j but G is false in W_j

$\Rightarrow (F \rightarrow G)$ is false in W_j

$\Rightarrow [](F \rightarrow G)$ is false in W_i - a **contradiction.**

Hence proved.

Theorems that hold in *K*

- $[](p \wedge q) \iff ([]p \wedge []q)$
- $([]p \vee q) \rightarrow [](p \vee q)$
- $[]p \iff \neg\Diamond(\neg p)$

T Structure

T is defined as a reflexive Kripke structure
or equivalently as
$T = K+$ T; T : $([]p \rightarrow p)$
Proof: TPT reflexivity $\rightarrow ([]p \rightarrow p)$.
Assume $([]p \rightarrow p)$ is not true in K(reflexive).
$\Rightarrow \exists$ a world $W_i s.t.$([]p $\rightarrow p)$ is not true in W_i.
$\Rightarrow$ in W_i $[]p$ is true and p is false.
But since $[]p$ is true in W_i and $(W_i, W_i) \in$ R (due to reflexivity),
p is true in W_i. — a Contradiction.
Hence proved.

The proof of $([]p \rightarrow p) \rightarrow reflexivity$ is left as an exercise.

Theorems that hold in *T*

- $p \rightarrow \Diamond p$
- $\Diamond(p \rightarrow []p)$

S4 Structure

S4 is defined as a reflexive and transitive Kripke structure or equivalently as

$S4 = T+$ S4 ; S4 : $([]p \rightarrow [][]p)$

Proof: TPT $reflexivity \rightarrow ([]p \rightarrow p)$

Assume $([]p \rightarrow [][]p)$ is not true in K(reflexive and transitive).

$\Rightarrow \exists$ a world W_i s.t. $([]p \rightarrow [][]p)$ is not true in W_i.

$\Rightarrow$ in $W_i []p$ is true and $[][]p$ is false.

$\Rightarrow \exists W_j$ s.t.$(W_i, W_j) \in$ R and $[]p$ is false in W_j.

$\Rightarrow \exists W_k$ s.t. $(W_j, W_k) \in$ R and p is not true in W_k.

But since $(W_i, W_k) \in R$ (due to transitivity) and $[]p$ is true in W_i, p is true in W_k. — a Contradiction.

Hence proved.

Theorems which hold in S4

- $\Diamond p \iff \Diamond\Diamond p$

S5 Structure

S5 is defined as a reflexive, transitive and symmetric Kripke structure or equivalently as
$S5 = S4+$ S5; S5 : $(\neg[]p \rightarrow [](\neg[]p))$
The proof of the above equivalence is left as an exercise.

The above axiom S5 can also be written as:
$(\Diamond p \rightarrow []\Diamond p)$

In S5 every modality is equivalent to one of the following or their negatives.

- $-, [], \Diamond$