CS206 Lecture 19

Predicate Logic Semantics

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Plan for Lecture 19

- Why Predicate Logic?
- Domains, Interpretations, Models

Symbolic Logic

Two components of symbolic logic are:

1. A language for representing statements and arguments. This language provides a precise medium/language for expressing world knowledge. This language has two aspects: Syntax and Semantics/Interpretation.
2. A means for 'manipulating' logical statements - Deduction/Axiomatic System.

Symbolic Logic can be broadly classified into four subclasses:

1. Propositional Logic.
2. First Order Logic or Predicate Calculus.
3. Higher Order Logics.
4. Modal Logics.

This classification is based upon the expressive power of the logic.

First Order Logic

First order logic is more powerful than Propositional Logic.
For example, consider the following standard arguments.

> *Every* man is mortal. Chanakya is a man. *Therefore* Chanakya is mortal.

is purely logical but can not be expressed in Propositional Logic.

> *Every* IITian stays in the campus, and
> Ajay is an IITian.
> *Hence*, Ajay stays in the campus.

> 5 is a prime number and it is odd.
> Therefore, *there exists* an odd prime number.

From Propositions to Predicates

Predicate Logic has the following richer view of declarative statements:

- Declarative statements are statements asserting that certain properties hold for some, all or particular objects called individuals.
- Complex declarative statements can be formed by quantifying (like for all individuals) or *there exists individuals* satisfying some property.

Beyond Predicate Logic

FOL though more expressive than PL can not express modalities like belief, tense as the following examples show:

- I am a student today but will not be a student in a few years.
- God exists.
  I believe that God exists.
  I know that God exists.
- Whenever it rains for more than a day, there is a flood the for next three days.
- It rains every week, except in summers.

Predicate Logic Syntax

Alphabet uses following sets.

- $C = \{a, b, ...\}$ a countable set of constant symbols.
- $F = \{f, g, ...\}$ a countable set of function symbols.
- $P = \{p, q, ...\}$ a countable set of predicate symbols.
- $V = \{x, y, ...\}$ a countable set of variables.
- $Conn = \{\neg, \wedge, ...\}$ is the set of connectives.
- $Val = \{\mathbf{t}, \mathbf{f}\}$ is the set of truth constants.
- $Q = \{\forall, \exists\}$ is the set of quantifiers.

Well Formed Formulae

Atomic Formulae

1. The truth constants $\mathbf{t}$, $\mathbf{f}$ are atomic forumlae.
2. If p is a predicate symbol of arity n and $t_1, ..., t_n$ are terms, then $p(t_1, ..., t_n)$ is an atomic formula.
3. Nothing else is an atomic formula.

Well Formed Formulae

1. Every atomic formula is a wff.
2. If ϕ_1 and ϕ_2 are wff, then so are
   - $\neg(\phi_1)$
   - $\phi_1 \wedge \phi_2$
   - $\phi_1 \vee \phi_2$
   - $\ldots$
3. If ϕ is a wff and x a varibale, then $\forall x\ \phi$ and $\exists x\ \phi$ are wff.
4. Nothing else is a wff.

Examples of WFFs

1. $(p(X,Y) \wedge (q(X) \vee \neg F))$
2. $(p(X,Z) \vee (q(Y) \wedge T)) \vee (\forall X p(X,Z) \rightarrow q(Y))$.
3. $(\neg(\exists X p(X) \vee \forall Y q(Y)) \leftrightarrow \neg \forall X, Y (p(X) \vee \neg q(Y)))$

Some examples of Non-WFFs are $\forall X \wedge q$,$\forall p, f \;:\; p(X, f(Y))$. and $\forall X f(X)$.

Scope, Free, Bound of Variables

The *scope of a quantifier* is the sub-formula over which the quantification is applicable.

Bound and Free variables: An occurrence of a variable is *bound* if that occurrence lies within the scope of a quantifier quantifying that variable or is the occurrence in that quantifier. In contrast, an occurrence is *free* if it is not within the scope of any quantifier quantifying that variable. These notions can be extended easily to variables themselves: A variable is *free in a formula* if there is at least one free occurrence of the variable in the formula; a variable is *bound in a formula* if there is at least one bound occurrence of the variable in the formula.

Examples

To illustrate these notions, consider the formula
$\exists Z : (\forall X(p(X) \wedge q(X)) \wedge \exists X r(X)) \rightarrow \forall X T(f(X), Z)$
Here Z in $T(f(X), Z)$ is not in the scope of the quantification and hence a free variable in the formula. On the other had all the X's are bound, but by three different quantifications.
Closed formulae: A wff is said to be **closed** if it does not contain a free variable. For example, the formula
$\forall Z : ((\forall X : p(X, Z)) \rightarrow \exists X : p(X, Z))$
is closed.

Semantics of FOL

As in PL, the semantics or interpretation is abstract as it assigns truth values to wffs. An interpretation of a wff consists of

- A nonempty domain D and an assignment of values to each individuals, function and predicate symbols occurring in the formula as follows:
  - To each individual and variables, an element of D is assigned.
  - To each n-ary function symbol, a mapping from $D^n \longrightarrow D$ is assigned.
  - To each n-ary predicate symbol, an n-ary relation over D is defined.

Truth over the Domain

Given such an interpretation, a wff is assigned a truth value as follows:

- If the subformulae G and H are assigned truth values then the truth values for the formulae $\neg G, (G \wedge H), (G \vee H), (G \rightarrow H), (G \leftrightarrow H)$ are evaluated using the truth tables (propositional logic) for these operators.
- $\forall X G$ has the truth value $true$ iff G is evaluated to $true$ for each d in D.
- $\exists X G$ has the truth value $true$ iff G is evaluated to $true$ for at least one d in D.

An interpretation with domain D is called an *interpretation over* D.

Interpretations

More than one interpretation is possible for a formula which arise out of different choices of D and different interpretation of symbols over a given D.

Example:

Consider the two formulae $forall X \exists Y p(X, Y)$ and $\exists Y \forall X p(X, Y)$. We have a number of interpretations possible for these formulae:

- Consider the interpretation:
  $D = \{0, 1, 2, \ldots\}$.
  $p(X, Y)$ is $X \geq Y$.
  Both the wff are true.

- Here is another interpretation:
  $D = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
  $p(X, Y)$ is as before.
  The first wff is true while the second one is false!

Models

An interpretation of a wff is called its *model* if the wff is true under that interpretation. An interpretation of a wff is called its *counter-model* if the wff is false under that interpretation. A wff is *valid* provided it is true under *all* interpretations. Examples of valid formulae:

1. $\forall X p(X) \rightarrow \exists X p(X)$.
2. $\forall X p(X) \rightarrow p(a)$.
3. $\exists Y \forall X p(X, Y) \rightarrow \forall X \exists Y p(X, Y)$.
4. $\exists X p(X, X) \rightarrow \exists X \exists Y p(X, Y)$.
5. $(\forall X p(X) \vee \forall X q(X)) \rightarrow \forall X p(X) \vee q(X)$.
6. $\exists X (p(X) \wedge q(X)) \rightarrow \exists X p(X) \wedge \exists X q(X)$.
7. $(\exists X p(X) \rightarrow \forall X q(X)) \rightarrow \forall X (p(X) \rightarrow q(X))$.

Examples of Invalid Formulae:

1. $\exists X p(X) \rightarrow \forall X p(X)$.
2. $p(a) \rightarrow \forall X p(X)$.
3. $\forall X \exists Y p(X,Y) \rightarrow \exists Y \forall X p(X,Y)$.
4. $\exists X \exists Y p(X,Y) \rightarrow \exists X p(X,X)$.
5. $\forall X (p(X) \vee q(X)) \rightarrow (\forall X p(X) \vee \forall X q(X))$.
6. $\exists X p(X) \wedge \exists X q(X) \rightarrow \exists X (p(X) \wedge q(X))$
7. $\forall X (p(X) \rightarrow q(X)) \rightarrow (\exists X p(X) \rightarrow \forall X q(X))$.

Satisfiability

- A wff is satisfiable provided it is true under some interpretation, i.e. there exists a model.

  Note that all valid wffs are satisfiable, while some invalid ones are satisfiable.

- A wff is a contradiction or unsatisfiable if and only if it is false under all interpretations.

  Therefore a negation of valid formula is unsatisfiable.

Some Important Equivalences in FOL

- All PL equivalences hold in FOL.
- Duality of Quantifiers:
  - $\neg \forall X A(X) \leftrightarrow \exists X \neg A(X)$.
  - $\neg \exists X A(X) \leftrightarrow \forall X \neg A(X)$.
- Scope inclusion/exclusion rules: The following set of equivalences and their symmetric counterparts are all valid:
  - $\exists X A(X) \vee B \leftrightarrow \exists X(A(X) \vee B)$.
  - $\forall X A(X) \vee B \leftrightarrow \forall X(A(X) \vee B)$.
  - $\exists X A(X) \wedge B \leftrightarrow \exists X(A(X) \wedge B)$.
  - $\forall X A(X) \wedge B \leftrightarrow \forall X(A(X) \wedge B)$.

  where X does not occur free in B.