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CS206 Lecture 20

Predicate Logic Semantics

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Fri, Feb 28, 2003

Plan for Lecture 20

- Domains, Interpretations, Models
- Examples

Semantics of FOL

Semantics (or interpretation) assigns truth values to wffs.

An interpretation of a wff consists of

- A nonempty (countable) domain D.
- An assignment of values to each individuals (constants and free variables), function and predicate symbols occurring in the formula as follows:
  - To each constant and free variable, some element of D is assigned.
  - To each n-ary function symbol, a mapping from $D^n \longrightarrow D$ is assigned.
  - To each n-ary predicate symbol, an n-ary relation over D is defined.

Truth over the Domain

Given such an interpretation, a wff is assigned a truth value as follows.

- Atomic Forumlae ($P(t_1, ..., t_n)$) is looked up in the interpretation.
- If the subformulae G and H are assigned truth values then the truth values for the formulae $\neg G, (G \wedge H), (G \vee H), (G \rightarrow H), (G \leftrightarrow H)$ are evaluated using the truth tables (propositional logic) for these operators.
- $\forall X G$ has the truth value $true$ iff G is evaluated to $true$ for each d in D.
- $\exists X G$ has the truth value $true$ iff G is evaluated to $true$ for at least one d in D.

Interpretations

More than one interpretation is possible for a formula which arise out of different choices of D and different interpretation of symbols over a given D.

$p(a)$

- Interpretation: $D = \{1, 2\}$.
  Assignment for a is 1.
  Assignment for p is $p(1) = true$ and $p(2) = false$.
  Under this interpretation the given wff is true.
- Another Interpretation: D and p as before, but a is 2.
  The the wff is false under this interpretation.

$\forall X p(X)$ **and** $\exists X p(X)$**:**

- Under both the above interpretations, the first wff evaluates to false while the second one evaluates to true.
- Consider other interpretations:
  $D = \{0, 1, 2, \cdots\}$, $p(X)$ is the relation X is odd.

Yet Another Example

$\forall X p(X, f(X))$:

- An interpretation that satisfies this formula is:
  Domain: $D = \{0, 1, 2, \cdots, \}$,
  Function: $f(X) = x + 1$,
  Predicate: $p(X, Y)$ is true iff $X \leq Y$.
- Another interpretation (Herbrand Interpretation):
  Domain: $D = \{a, f(a), f(f(a)), \cdots, \}$,
  Function: $f(a) = f^1(a)$,
  $f(f^n(a)) = f^{n+1}(a)$.
  Predicate: $p(X, Y)$ is true iff
  $x = f^n(a), y = f^m(Y)$ and $n \leq m$.

Interesting Example

Consider the two formulae $\forall X \exists Y p(X, Y)$ and $\exists Y \forall X p(X, Y)$. We have a number of interpretations possible for these formulae:

- Consider the interpretation:
  $D = \{0, 1, 2, \ldots\}$.
  $p(X, Y)$ is $X \geq Y$.
  Both the wff are true.

- Here is another interpretation:
  $D = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
  $p(X, Y)$ is as before.
  The first wff is true while the second one is false!

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Models

An interpretation of a wff is called its *model* if the wff is true under that interpretation. An interpretation of a wff is called its *counter-model* if the wff is false under that interpretation. A wff is *valid* provided it is true under *all* interpretations. Examples of valid formulae:

1. $\forall X p(X) \rightarrow \exists X p(X)$.
2. $\forall X p(X) \rightarrow p(a)$.
3. $\exists Y \forall X p(X, Y) \rightarrow \forall X \exists Y p(X, Y)$.
4. $\exists X p(X, X) \rightarrow \exists X \exists Y p(X, Y)$.
5. $(\forall X p(X) \vee \forall X q(X)) \rightarrow \forall X p(X) \vee q(X)$.
6. $\exists X (p(X) \wedge q(X)) \rightarrow \exists X p(X) \wedge \exists X q(X)$.
7. $(\exists X p(X) \rightarrow \forall X q(X)) \rightarrow \forall X (p(X) \rightarrow q(X))$.

Examples of Invalid Formulae:

1. $\exists X p(X) \rightarrow \forall X p(X)$.
2. $p(a) \rightarrow \forall X p(X)$.
3. $\forall X \exists Y p(X, Y) \rightarrow \exists Y \forall X p(X, Y)$.
4. $\exists X \exists Y p(X, Y) \rightarrow \exists X p(X, X)$.
5. $\forall X (p(X) \vee q(X)) \rightarrow (\forall X p(X) \vee \forall X q(X))$.
6. $\exists X p(X) \wedge \exists X q(X) \rightarrow \exists X (p(X) \wedge q(X))$
7. $\forall X (p(X) \rightarrow q(X)) \rightarrow (\exists X p(X) \rightarrow \forall X q(X))$.

Satisfiability

- A wff is satisfiable provided it is true under some interpretation, i.e. there exists a model.

  Note that all valid wffs are satisfiable, while some invalid ones are satisfiable.

- A wff is a contradiction or unsatisfiable if and only if it is false under all interpretations.

  Therefore a negation of valid formula is unsatisfiable.

Some Important Equivalences in FOL

- All PL equivalences hold in FOL.
- Duality of Quantifiers:
  - $\neg\forall X A(X) \leftrightarrow \exists X \neg A(X)$.
  - $\neg\exists X A(X) \leftrightarrow \forall X \neg A(X)$.
- Scope inclusion/exclusion rules: The following set of equivalences and their symmetric counterparts are all valid:
  - $\exists X A(X) \vee B \leftrightarrow \exists X (A(X) \vee B)$.
  - $\forall X A(X) \vee B \leftrightarrow \forall X (A(X) \vee B)$.
  - $\exists X A(X) \wedge B \leftrightarrow \exists X (A(X) \wedge B)$.
  - $\forall X A(X) \wedge B \leftrightarrow \forall X (A(X) \wedge B)$.

  where X does not occur free in B.

Encoding World-Knowledge

Every man is mortal. Chanakya is a man. *Therefore* Chanakya is mortal.

Every IITian stays in the campus, and
Ajay is an IITian.
Hence, Ajay stays in the campus.

5 is a prime number and it is odd.
Therefore, *there exists* an odd prime number.

1. $[\forall X(man(X) \rightarrow mortal(X)) \wedge man(Chanakya)] \rightarrow mortal(Chanakya).$
2. $[\forall X(iitian(X) \rightarrow campusite(X)) \wedge iitian(A)] \rightarrow campusite(A).$
3. $prime(5) \wedge odd(5) \rightarrow \exists X(prime(X) \wedge odd(X)).$

More Examples

- At least one hour is free.

  $\exists X freehour(X)$.

- A thing is a pen only if it writes and holds ink.

  $\forall X(write(X) \wedge ink(X) \rightarrow pen(X))$

- All that glitters is not gold

  $$\neg(\forall X glitter(X) \rightarrow gold(X))$$

  (Compare with: $\forall X(glitter(X) \rightarrow \neg gold(X))$)

  Alternatively: $\exists X(glitter(X) \wedge \neg gold(X))$

- For every positive number there is a smaller number.

  $\forall X \exists Y gt(X, Y)$

Validity of FOL formulae

The methods employed for determining validity of propositional formulae can not be directly extended.

- Truth table method of PL can not be extended as the truth table for FOL would require an infinite table!
- Normal Forms Method is also less effective since we can not have normal forms that can be syntactically checked to determine whether a wff is valid.

Normal forms are however, useful as they allow one to assume a fixed syntactic form for wffs. Two normal forms for FOL wffs are defined *Prenex Conjunctive Normal Form* and *Prenex Disjunctive Normal Form*.

Prenex Normal Forms

A wff is in prenex conjunctive normal form (PCNF) if

- it is either T or F or
- it is of the form $Q_1x_1 \cdots Q_nx_nM$, where each Q_ix_i is either $\forall x_i$ or $\exists x_i$ and M is a wff containing no quantifiers and is in conjunctive normal forms. $Q_1 \cdots Q_n$ is called *prefix* and M is called the *matrix*.

Examples of PCNF

- $\forall X \forall Y (p(X, Y) \wedge q(Y))$.
- $\forall X \exists Y (\neg p(X, Y) \vee q(X, Y))$.
- $\forall X \forall Y \exists Z ((\neg q(Y) \vee p(X, Y)) \wedge r(X, Z))$.

Exercise Design an algorithm for converting any wff to prenex normal form.