CS206 Lecture 21
Modal Logic

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Thu, Mar 13, 2003

Plan for Lecture 21

- Modal Logic
- Possible World Semantics

Modal Logic

- FOL is a big improvement over PL in terms of expressive power
- But, there are many arguments that cannot be expressed in FOL.
- Many facts in real world are dynamic and their truth is a relative notion.

Consider the following.

- Either it rains or it does not rain.
- It may rain today.
- It rains sometimes.
- It always flooded after every rains.
- I believe that I may be wrong.
- I believe that Ram knows that I know that he did it.

The first statement is true absolutely in all situations, at all places and at all times. The truth value of the rest depends on the place, time and the judgment of the person who uttered it.

Bindi Puzzle

A mother wishes to test her three daughters. She arranges them in a circle so that they can see and hear each other and tells them that she will put a white or black bindi on each of their foreheads but that at least one bindi will be white. In fact all three bindis are white. She then repeatedly asks them, "Do you know the color of your bindi?" What do they answer?

Sum and Product Puzzle

Two numbers m and n are chosen such that $2 \leq m, n \leq 99$. Mr. S is told their sum and Mr. P is told their product. The following dialogue ensues:

- Mr. P: I don't know the numbers.
- Mr. S: I knew you didn't know. I don't know either.
- Mr. P: Now I know the numbers.
- Mr S: Now I know them too.

So, do you know the numbers now?

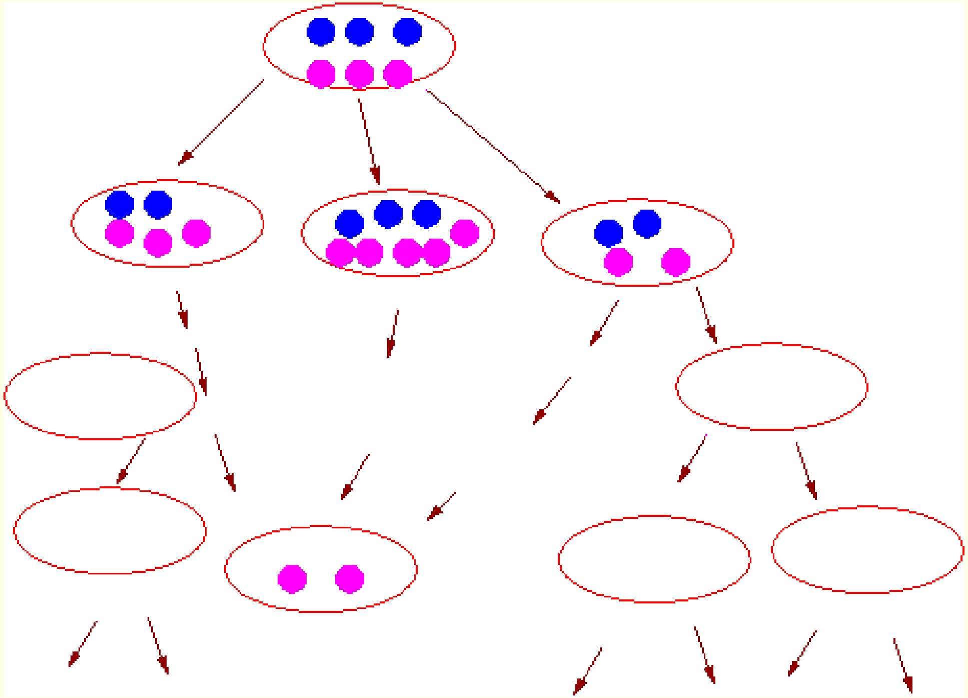
Another Example

Consider an island starting initially with n male rabbits and n female rabbits. Assume the following things can happen any time.

- 2 males can fight with each other and one dies.
- A male and a female can produce one of the following litters
  - 1 male and 2 female
  - 1 male and 1 female
  - 2 females
- An (old?) couple (1 male and 1 female) commit suicide together.

Can translate above easily into some problem on vlaues of variables in a program (but that takes away the fun?)

Possible Scenarios (Worlds)



Some Theorems

How can we state and prove/refute the following?

- There will always be at least as many females as males.
- Whenever there is at least one male left, it is possible for the entire species to become extinct.
- So long as at least one male and one female survive, it is possible that the number of males will become a prime number and the number of females will be twice this.
- ...

Which ones are

- **Necessary Truths?**
- **Possible Truths?**

MODAL LOGIC

Modal logics have been developed precisely to express these kinds of statements. Modal logic once a subject of philosophy, has now become a branch of mathematical logic. Nowadays computer scientists use modal logic to reason about knowledge, time, beliefs and proof theory. The simplicity, elegance and power of modal logic can be best seen in its application in reasoning about proofs, where it throws light on Gödel's theorems.

Modal logic is an extension of propositional logic by introducing modalities on propositions: instead of a proposition being merely just true or false, it may in addition be *necessarily* true or *possibly* true. This investigation was largely confined to the domain of philosophical logic. However in the 1950s Kripke gave precise mathematical meanings to the notions of modality in terms of possible world models and brought it in the domain of mathematics. Besides having precise syntax and semantics, the logic was shown to have several applications by appropriately interpreting the modalities.

Modalities

The two modalities *necessity* and and its dual *possibility* can qualify formulae with several useful interpretations:

Logic	$\Box$ (G)	$\Diamond$ (F)
Modal	Necessary	Possible
Epemistic	Knowledge	Belief
Deontic	Obligation	Permitted
Proof Theory	Provable	Consistent
Temporal	Always	Sometimes

Can we do all this in First-order logic?

It is possible to do some modal logic in FOL by means of interpretations. For example, by using the natural numbers to represent the time points and using predicates to assert their truth values at different time points. However, the modalities let us get down directly to the issues of temporal properties, instead of first having to define the properties of time points using the axiomatization of natural numbers.
Hence the modalities let us concentrate on the issues at hand, and we need to only define the properties of the modalities.

Duality

$$\Box\phi \leftrightarrow \neg\Diamond\neg\phi$$

$$\Diamond\phi \leftrightarrow \neg\Box\neg\phi$$

Example:

- It sometimes rains.
- It is not the case that it is always dry.

$$\Diamond Rain \leftrightarrow \neg\Box\neg Rain$$

Examples of Modalities in Use

- I believe '2 + 2 = 4' is true: $B(2+2=4)$.
- It always rains and sometimes floods here: $\Box Rain \wedge \Diamond flood$.
- The formula ϕ is neither provable nor refutable: $\neg\Box(\phi) \wedge \neg\Box\neg\phi$.
- It is possible that he knows that ϕ is true: $\Diamond K\phi$.

Possible Worlds semantics

In philosophical logic, the notion of absolute truth of a proposition was replaced by the notion of truth with respect to the *world* where it was uttered. It envisaged a set of worlds related to each other. A proposition was necessarily true in a world if it was true in every world related to that world, and possibly true in it if it was true in at least one world related to it.

The syntax

The *language* of propositional modal logic consists of logical and nonlogical symbols. The nonlogical symbols are a set of propositions which assert facts about the world.

The Alphabet

- The truth constants: true ($\mathcal{T}$), false ($\mathcal{F}$);
- The usual propositional connectives: conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), if and only if ($\leftrightarrow$), negation ($\neg$);
- And the modal operators: *necessarily* ($\Box$) and *possibly* ($\Diamond$).

Well formed formulae

The set of wffs is given by applying the following rules any number of times:

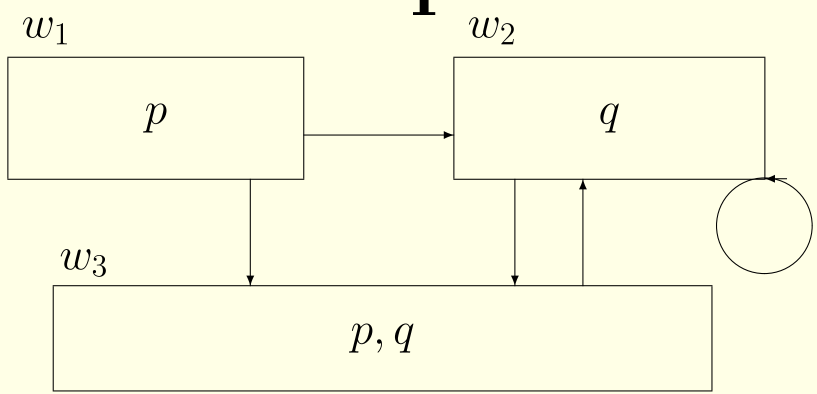
1. Every proposition is a wffs
2. If ϕ and ψ are wffs then so are the following.
   - $\phi \wedge \psi$
   - $\phi \vee \psi$
   - $\phi \longrightarrow \psi$
   - $\neg\phi$
   - $\Box\phi$
   - $\Diamond\phi$

Truth of formulae

Every formula must be either true or false in any given situation (*model*). Moreover the truth value is meaningfully assigned: for example, if both ϕ and ψ are assigned $\mathcal{T}$, then $(\phi \wedge \psi)$ will also be assigned $\mathcal{T}$. Using possible world as our basic model, truth value can be assigned to every formula of modal logic.

A possible world model consist of a set *worlds*, a binary (accessibility) relation between the worlds, and truth assignments to the propositions in every world. The formula $\forall\phi$ is true in a world w, if and only if ϕ is true in all worlds related to w. Whereas its dual formula $\exists\phi$ is true in a world w, if and only if ϕ is true in some (at least one) world related to w.

Example of a Model



There are three possible worlds:w_1, w_2 and w_3. Proposition p is true in w_1 and w_3. Proposition q is true in w_2 and w_3. The accessibility relation R is:

$$R(w_1, w_2), R(w_1, w_3), R(w_2, w_2), R(w_2, w_3), R(w_3, w_2).$$

Then the following modal formulae are true in the given world:

$$\begin{array}{ll} w_1 & \Box q, \Box\Box q, \Diamond p, \Diamond\neg p. \\ w_2 & \Box q, \Diamond p, \Diamond\neg p. \\ w_3 & \Box q, \Box\neg p, \Diamond q. \end{array}$$