# CS217: Artificial Intelligence and Machine Learning (associated lab: CS240)

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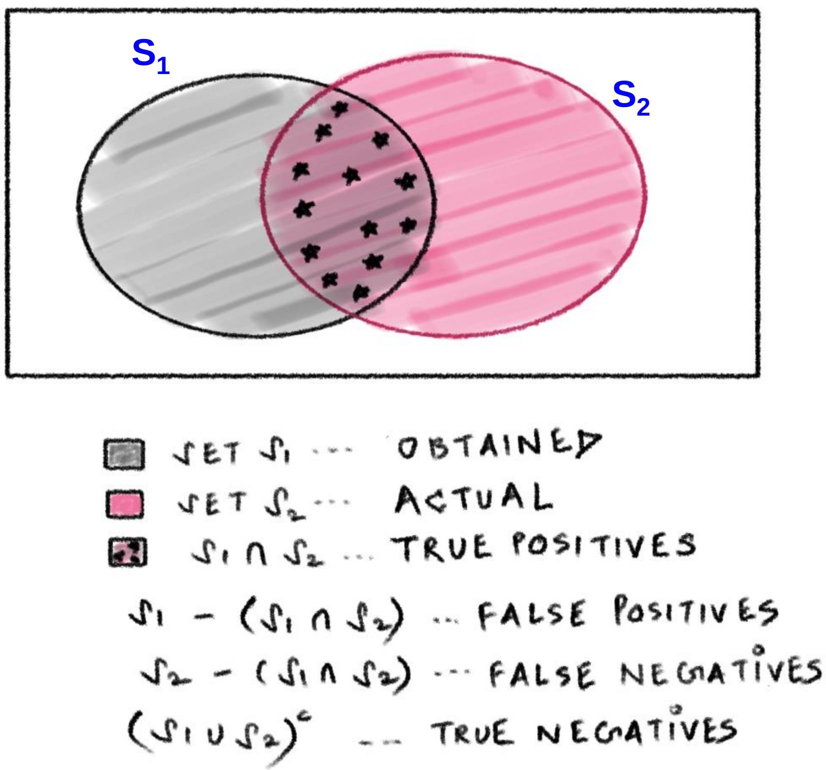
*Week10 of 17mar25, Linear, logistic regression, Decision Trees*

# Main points covered: week9 of 10mar25

# SVM using SKLEARN

- Dataset from the UCI Machine Learning Repository
  - https://archive.ics.uci.edu/ml/datasets.php
- Number of instances = 748
- Number of attributes = 4
- **Attributes / Features**:
  - Recency - months since last donation
  - Frequency - total number of donation
  - Monetary - total blood donated in c.c.
  - Time - months since first donation
- **Class label**:
  - A binary variable representing whether he/she donated blood in March 2007
  - 1 stands for donating blood; 0 stands for not donating blood

# False Positives, False Negatives, Precision, Recall, F-score




$$Precision = \frac{|S_1 \bigcap S_2|}{|S_1|}$$

$$Recall = \frac{|S_1 \bigcap S_2|}{|S_2|}$$

# Generalized F-score

$$F_\beta = \frac{(1+\beta^2)PR}{\beta^2 P + R} = \frac{1}{\frac{\beta^2}{(1+\beta^2)R} + \frac{1}{(1+\beta^2)P}}$$

As $\beta \rightarrow 0$, $F_\beta \rightarrow P$ and as $\beta \rightarrow \infty$, $F_\beta \rightarrow R$

# Formal definition of HMM (1/2)

- $N$= #states
- $M$= #distinct observation symbols
- State transition probability distribution: $A=\{a_{ij}\}$
- The observation symbol probability distribution in state $j$, $B=b_j(k)$
- The initial state distribution, $\pi=\{\pi_i\}$

# Formal definition of HMM (2/2)

- $A=\{a_{ij}\}$
  - $a_{ij}= P(q_{t+1}=S_j \mid q_t=S_i),\ 1<= i,j <=N$
- $B=b_j(k)$
  - $b_j(k)= P(V_k \text{ at } t \mid q_t= S_j),\ 1<= j <=N;\ 1<= k <= M$
- $\pi=\{\pi_i\}$
  - $\pi_i = P(q_1=S_i),\ 1<= i<=N$

# Classic problems with respect to HMM

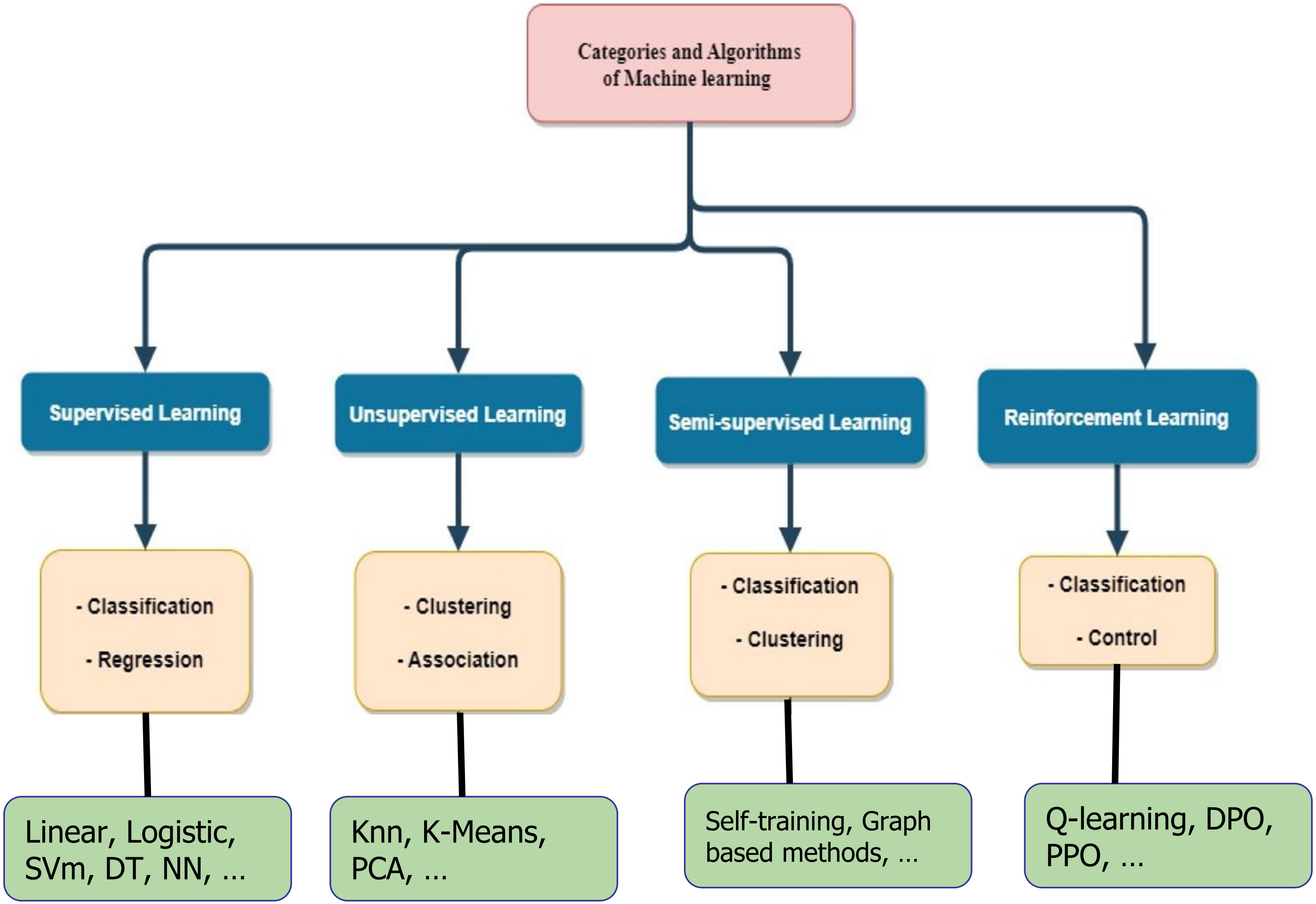
1. Given the observation sequence, find the possible state sequences- **Viterbi**

2. Given the observation sequence, find its probability- **forward/backward** algorithm

3. Given the observation sequence find the HMM prameters.- **Baum-Welch** algorithm

End of Main points

# General Machine Learning Tasks (1/2)




Others: Self-supervised learning, Active learning, Curriculum learning, Life-long learning

# General Machine Learning Tasks (2/2)

### Supervised

```
   Training Data
 (X, y pairs)
        |
        ▼
[ Model Training ]
        |
        ▼
 [ Learned Model ]
        |
        ▼
   New Input (X')
        |
        ▼
   [ Prediction ]
        |
        ▼
   Output (y')
```

### Unsupervised

```
              Input Data (X)
                    |
                    ▼
         [ Pattern Discovery ]
                    |
                    ▼
[ Clustering / Dimensionality Reduction ]
                    |
                    ▼
   Output: Patterns or Groupings
```

### Semi-supervised

```
Labeled Data (X_l, y_l)      Unlabeled Data (X_u)
        |                          |
        └────────────┬─────────────┘
                     |
                     ▼
      [ Semi-Supervised Learning ]
                     |
                     ▼
            [ Learned Model ]
                     |
                     ▼
             New Input (X')
                     |
                     ▼
           [ Prediction (y') ]
```

### Reinforcement

```
      [ Environment ]
            ▲
(Reward/Punishment)
            |
        [ Agent ]
            |
      (Action/Policy)
            ▼
      [ Environment ]
```

# Classification vs. Regression

**Classification**: a supervised learning task where the objective is to **predict discrete labels or categories** to input data. The model learns to map input features to one of a limited number of classes.

**Examples**: image recognition, text classification, etc.

**Regression**: a supervised learning task where the goal is to **predict continuous numeric values**. The model learns the relationship between input features and a continuous output.

**Examples**: stock market prediction, weather forecasting, etc.

# Linear Regression: Intro

Assumes *Linear relationship* between dependent (y) and independent (x) variables.

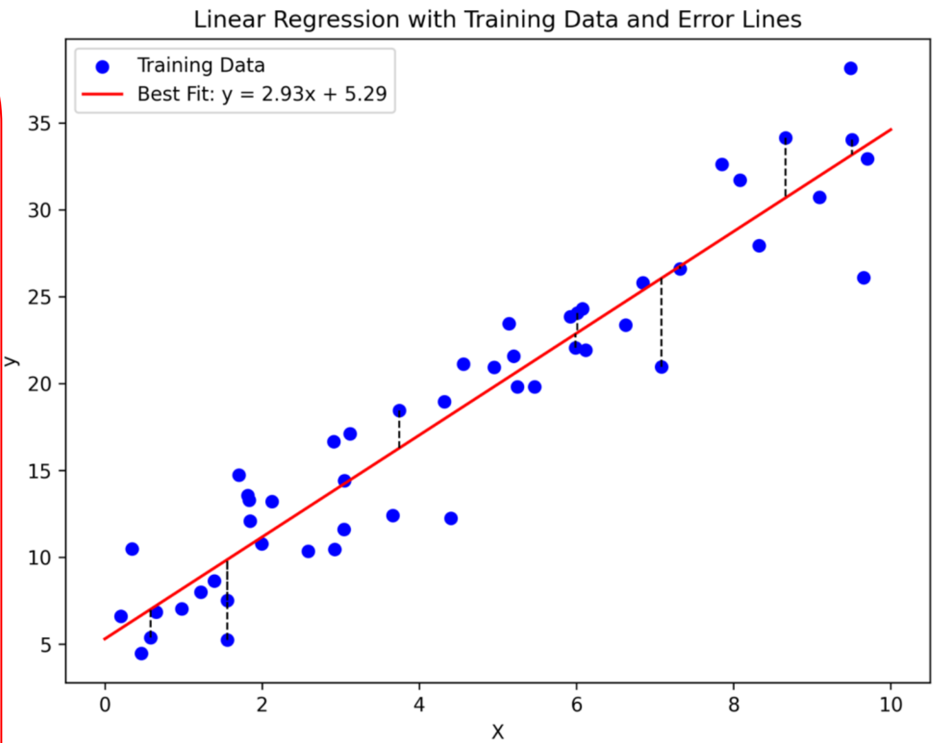
$$\hat{y} = f_\theta(x)$$

Here, *f* is the model with parameter $\theta$ that captures the linear relation between input *x* to output *y*.

$$y = mx + C + \epsilon$$

where $\epsilon \sim N(0, \sigma^2)$

1. Here, $\varepsilon$ is the error (random noise) caused due to experimental design.
2. b is the intercept.
3. m is the slope (indicating the change in *y* for a one-unit change in *x*)
4. $y$ is ground truth
5. $\hat{y}$ is predicted output




Assumes the dependent variable (y) is continuous in nature.

Simple linear regression: $y = b + w_1x + \epsilon$

Multiple linear regression: $y = b + w_1x_1 + w_2x_2 + \cdots + w_px_p + \epsilon$

# Linear Regression: **Setup and Error**

Given a dataset of input-output pairs, $\{(x_i, y_i)\}_{i=1}^{N}$ .

We train a linear model $f(x; w, b) \Rightarrow w^T x + b$, where $w$ and $b$ are model parameters.

Loss function: Sum Squared Error (SSE) => L(w, b) = $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - (b + wx_i))^2$

Goal: To find the model parameters that minimizes the loss

$$\arg\min_{w,b} L(w,b) = \arg\min_{w,b} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \arg\min_{w,b} \sum_{i=1}^{n} (y_i - (b + wx_i))^2$$

Compactly we car write: $\hat{y} = w^\top x, \text{ where } w = \begin{pmatrix} w_1 \\ \vdots \\ w_p \\ b \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 1 \end{pmatrix}$

Assumptions

1. **Linearity**: The relationship between predictors and response is linear
2. **Independence**: Observations are independent of each other
3. **Homoscedasticity**: Error variance is constant across all levels of predictors
4. **Normality**: Errors are normally distributed
5. **No multicollinearity**: Predictor variables are not highly correlated

# Least Square via **Calculus**

We know, L(w, b) = $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - (b + wx_i))^2$

Compute partial derivatives with respect to *b* and *w* and set them to zero:

For *b*: $\frac{\partial L}{\partial b} = -2 \sum_{i=1}^{n} (y_i - (b + wx_i)) = 0$ $\frac{\partial L}{\partial w} = -2 \sum_{i=1}^{n} x_i (y_i - (b + wx_i)) = 0$

Solving the above equations will give us:

$$w = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$b = \bar{y} - w\bar{x}$$

This is the solution for simple linear regression.

# Multivariate Least Square via **Calculus (1/2)**

Considering the loss function in compact form

$$L(w) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \left(y_i - w^\top x_i\right)^2 = \|y - Xw\|_2^2$$

Let $X = \begin{pmatrix} x_1^\top \\ \vdots \\ x_n^\top \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. We can rewrite it in terms of matrices:

$$\begin{aligned}
L(w) &= \|y - Xw\|_2^2 \\
&= (y - Xw)^\top (y - Xw) \\
&= y^\top y - (Xw)^\top y - y^\top (Xw) + (Xw)^\top (Xw) \\
&= y^\top y - 2y^\top (Xw) + \left(w^\top X^\top\right)(Xw) \\
&= y^\top y - \left(2y^\top X\right) w + w^\top \left(X^\top X\right) w
\end{aligned}$$

# Multivariate Least Square via **Calculus (2/2)**

$$\frac{\partial L}{\partial w} = \frac{\partial\left(y^\top y\right)}{\partial w} - \frac{\partial\left(\left(2X^\top y\right)^\top w\right)}{\partial w} + \frac{\partial\left(w^\top\left(X^\top X\right)w\right)}{\partial w} = 0$$

$$0 - 2X^\top y + \left(X^\top X + \left(X^\top X\right)^\top\right)w = 0$$

$$-2X^\top y + 2\left(X^\top X\right)w = 0$$

$$2\left(X^\top X\right)w = 2X^\top y \quad \text{(Normal equations)}$$

If $X$ is full-rank,

$$w = \left(X^\top X\right)^{-1} X^\top y$$

This also called the **Closed-form solution** to Linear Regression or Ordinary Least Square regression.

If this is the closed-form solution then $w = \left(X^\top X\right)^{-1} X^\top y$ should be the global minimum. To prove this, we need to show that ***L(w)* is convex**.

# Convexity of Sum Square Error

Theorem: Hessian of a convex function is Positive Semi-definite.

Recall: $\frac{\partial L}{\partial w} = -2X^\top y + 2\left(X^\top X\right) w$

$$\frac{\partial^2 L}{\partial w \partial w^\top} = \frac{\partial}{\partial w}\left(\frac{\partial L}{\partial w}\right)$$

$$\begin{aligned}
&= \frac{\partial}{\partial w}\left(-2X^\top y + 2\left(X^\top X\right) w\right) \\
&= \frac{\partial}{\partial w}\left(-2X^\top y\right) + \frac{\partial}{\partial w}\left(2\left(X^\top X\right) w\right) \\
&= 0 + 2\left(X^\top X\right)^\top \\
&= 2X^\top X \\
&\succcurlyeq 0
\end{aligned}$$

Since the Hessian is always positive semi-definite, the loss function is convex.
So any critical point is a local and global minimum, so the critical point $w = \left(X^\top X\right)^{-1} X^\top y$ minimizes the loss function.

# Scenarios of OLS (based on Pseudoinverse)

The matrix $(X^T X)^{-1} X^T$ is known as the **pseudoinverse** of $X$, and is usually denoted as $X^\dagger$.

Case 1:

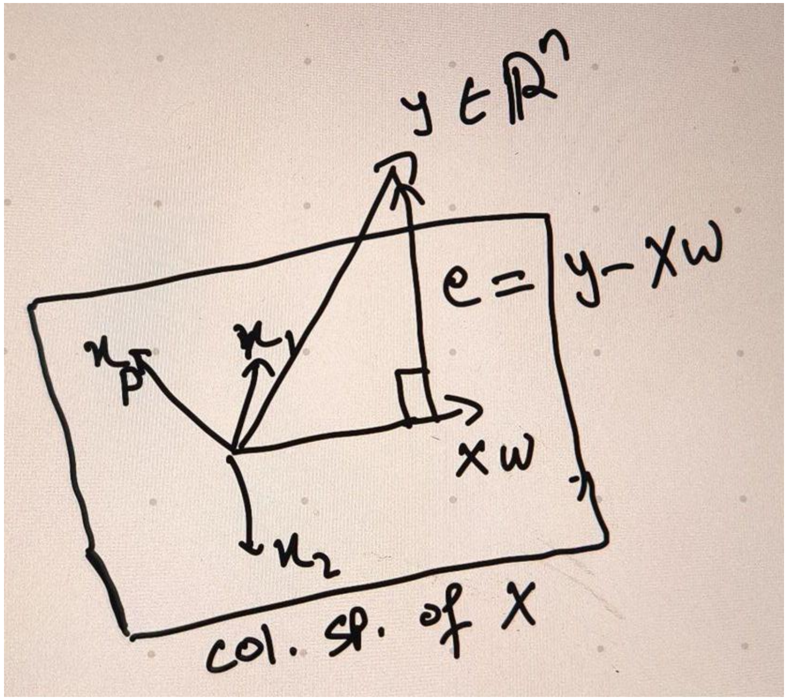
- When # of data points (N ) = # of input dimensions (p), can fit data perfectly.
  - In other words, $w = \min_w \|y - Xw\|_2^2$, so finding the optimal $w$ amounts to solving the linear system of equations $Xw = y$.
  - If X is full-rank, $w = X^{-1}y$.

Case 2:

- When # of data points (N ) > # of input dimensions (p), in general cannot fit data perfectly. (In real world scenarios, this is more practical)
  - As shown earlier, $w = (X^T X)^{-1} X^T y = X^\dagger y$. If we compare these formulas $w = X^{-1}y$ vs. $w = X^\dagger y$, $X^\dagger$ can be seen as a generalization of $X^{-1}$ to the case where data cannot be fit perfectly.
  - $X^\dagger$ can be interpreted as an operator that finds the best possible solution to a linear system of equations that has no solution

# Least Square via **Linear Algebra**

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_p \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$




In real world scenarios, n >> p

=> no solution generally

$y = Xw \Rightarrow$ (linear comb. of col of X )

col. sp (X) $\subseteq R^n$, dim(col(X)) $\leq$ p

$x_1^T e = 0, x_2^T e = 0, \cdots, x_p^T e = 0 \;\; \Rightarrow X^T e = 0$

$\Rightarrow x^T(y - Xw) = 0 \Rightarrow X^T X w = X^T y$

$\Rightarrow$ $\boldsymbol{w = (X^T X)^{-1} X^T y}$ (Same closed-form solution)

If Rank(X) = p.

$\Rightarrow$ Rank($X^T X$) = Rank(X) = p

$\Rightarrow$ Then Pseudo inverse can give solution

# Least Square (Closed Form): **Concerns**

1. High computational cost

   Computing $(X^T X)^{-1}$ requires matrix inversion, which generally has a computational cost on the order of $O(p^3)$, where $p$ is the number of features.

1. Numerical instability

   If the features are highly correlated (i.e., multicollinearity), $X^T X$ can be nearly singular. This makes the inversion numerically unstable, leading to unreliable coefficient estimates.

1. May require large memory

   When dealing with large-scale data, storing the entire $X$ matrix and computing $X^T X$ can exceed available memory.

# Least Square via **Gradient Descent**

$$w = \left(X^\top X\right)^{-1} X^\top y$$

This set of parameters minimizes the ″sum-squared-error” or ”mean-squared-error”

Remember, $SSE = \|Y - Xw\|_2^2$

$$\nabla L = \begin{bmatrix} \frac{\partial}{\partial w_1} L \\ \frac{\partial}{\partial w_2} L \\ \vdots \\ \frac{\partial}{\partial w_{p+1}} L \end{bmatrix} = -2X^\top y + 2(X^\top X)w$$

The gradient descent algorithm: Set $w^0$ to a random vector in $\mathbb{R}^{p+1}$ for an initial guess and choose a learning rate parameter $\gamma$. Compute $X^\top y$ (an element of $\mathbb{R}^{p+1}$ and $X^\top X(a\ (k+1) \times (k+1)$ matrix ).

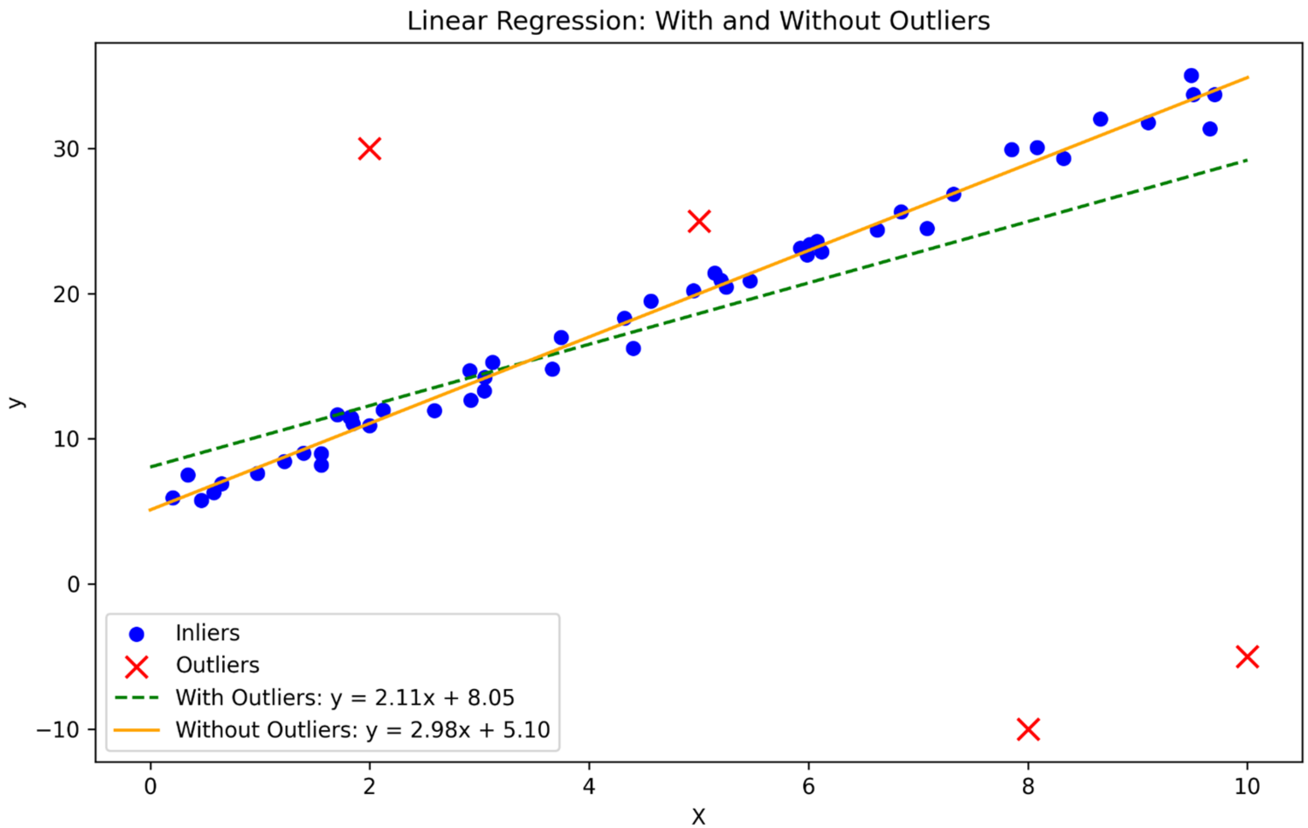
Iteratively compute

$$w^{(t+1)} = w^{(t)} - \gamma \left(-2X^\top y + 2(X^\top X)w^{(t)}\right)$$

until the entries a stopping condition is met. For example, stop if the mean squared error $\left\|y - Xw^{(k)}\right\|^2$ changes by less than some tolerance on each iteration, or the entries of $w^{(k)}$ change by less than some tolerance.

Notice that this algorithm does not need computation of $(X^\top X)^{-1}$.

# Least Square: **Handling Outliers (1/3)**




Linear regression: $L(w) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \left(y_i - w^\top x_i\right)^2 = \|y - Xw\|_2^2$

Ridge regression: $L(w) = \sum_{i=1}^{n} \left(y_i - w^\top x_i\right)^2 + \lambda\|w\|_2^2 = \|y - Xw\|_2^2 + \lambda\|w\|_2^2$

# Least Square: **Handling Outliers (2/3)**

$$L(w) = \|y - Xw\|_2^2 + \lambda\|w\|_2^2$$
$$= (y - Xw)^\top (y - Xw' + \lambda w^\top w$$
$$= y^\top y - (Xw)^\top y + y^\top (Xw) + (Xw)^\top (Xw) + \lambda w^\top w$$
$$= y^\top y - \left(2y^\top X\right) w + w^\top \left(X^\top X\right) w + \lambda w^\top w$$

$$\frac{\partial L}{\partial w} = \frac{\partial \left(y^\top y\right)}{\partial w} - \frac{\partial \left(\left(2X^\top y\right)^\top w\right)}{\partial w} + \frac{\partial \left(w^\top \left(X^\top X\right) w\right)}{\partial w} + \frac{\partial \left(\lambda w^\top w\right)}{\partial w} = 0$$

$$0 - 2X^\top y + \left(X^\top X + \left(X^\top X\right)^\top\right) w + \lambda \left(I + I^\top\right) w = 0$$
$$-2X^\top y + 2\left(X^\top X\right) w + 2\lambda I w = 0$$
$$-2X^\top y + 2\left(X^\top X + \lambda I\right) w = 0$$
$$2\left(X^\top X + \lambda I\right) w = 2X^\top y$$
$$\left(X^\top X + \lambda I\right) w = X^\top y$$
$$w = \left(X^\top X + \lambda I\right)^{-1} X^\top y$$

# Least Square: **Handling Outliers (3/3)**

$$\text{Recall: } \frac{\partial L}{\partial w} = -2X^\top y + 2\left(X^\top X + \lambda I\right) w$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial w \partial w^\top} &= \frac{\partial}{\partial w}\left(\frac{\partial L}{\partial w}\right) \\
&= \frac{\partial}{\partial w}\left(-2X^\top y + 2\left(X^\top X + \lambda I\right) w\right) \\
&= \frac{\partial}{\partial w}\left(-2X^\top y\right) + \frac{\partial}{\partial w}\left(2\left(X^\top X + \lambda I\right) w\right) \\
&= 0 + 2\left(X^\top X + \lambda I\right)^\top \\
&= 2\left(X^\top X + \lambda I\right)
\end{aligned}$$

Claim: $X^\top X + \lambda I \succ 0$

Proof: $w^\top \left(X^\top X + \lambda I\right) w = w^\top \left(X^\top X\right) w + w^\top(\lambda I)w = (Xw)^\top(Xw) + \lambda w^\top w = \|Xw\|_2^2 + \lambda\|w\|_2^2$

For any $w \neq \vec{0}, \|w\|_2^2 > 0$

Since $\|Xw\|_2^2 \geq 0$ and $\lambda > 0, \|Xw\|_2^2 + \lambda\|w\|_2^2 > 0 \quad V w \neq \vec{0}$

# Least Square: **Concerns**

1. Linearity assumption

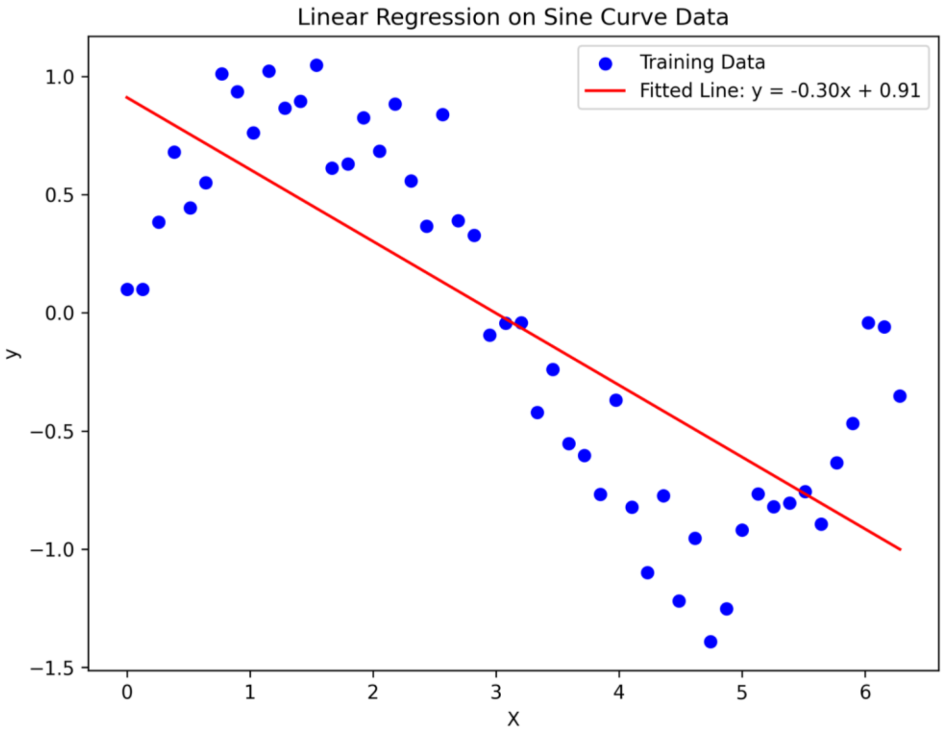
   What about non-linear relation between *X* and *y*?

- Kernel Trick

1. Sensitive to large outliers

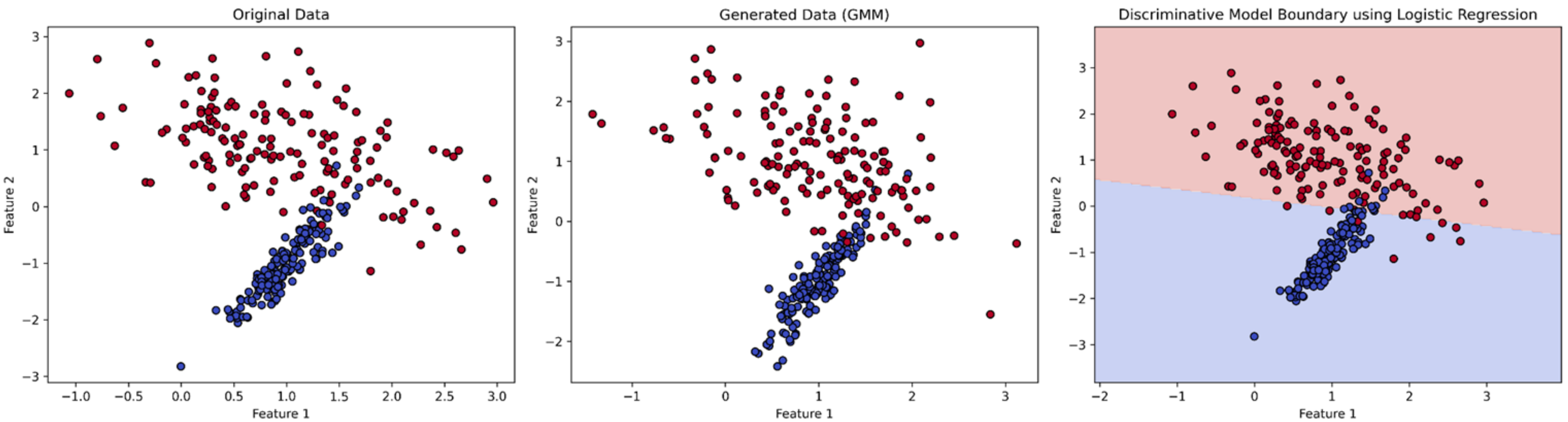
   It does not detect outliers, rather try to reduce the effect of outliers through regularization.

1. Less interpretable



# Least Square: **Code Walk through**

# **Discriminative** *vs.* **Generative** Model (1/2)




Generative Methods (***G***):
Generative models aim to capture how the data is generated by modeling the joint probability distribution p(x,y). In doing so, they "explain" both the features and the labels, which allows them not only to classify but also to generate new data samples.

A generative model is defined by: $p(x,y|\vartheta)=p(x|y,\vartheta)\,p(y|\pi)$

- p(y) is the prior distribution over the labels.
- p(x|y) is the likelihood, describing how the features x are generated given y.
- θ: the parameters of ***G***
- π: the class prior

# **Discriminative** *vs.* **Generative** Model (2/2)

Discriminative Methods (***D***):

Discriminative models focus directly on modeling the decision boundary between classes. Instead of modeling the joint distribution, they learn the conditional probability p(y|x) or a direct mapping from x to y. This often leads to better classification performance as the model is optimized solely for the prediction task.

- P(X,Y): the true joint distribution of inputs and outputs
- θ: the parameters of discriminative model
- ***D***: directly model the conditional probability distribution P(Y|X; θ).

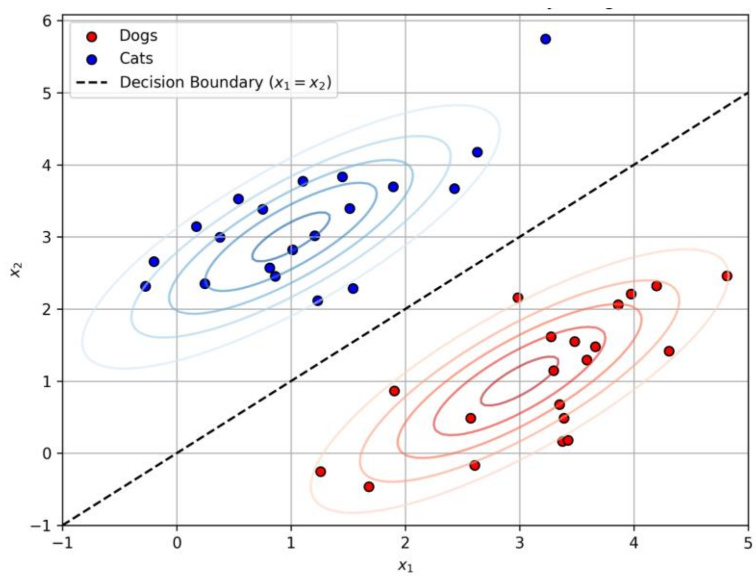
| **Aspect** | **Generative Methods** | **Discriminative Methods** |
|---|---|---|
| Modeling Objective | Models the joint distribution $p(x, y)$ | Models the conditional distribution $p(y\|x)$ or a decision function |
| Data Generation | Can generate new samples $x$ given a label $y$ | Focus solely on predicting $y$ given $x$ |
| Assumptions | Often require assumptions about data generation (e.g., conditional independence in Naive Bayes) | Fewer assumptions about how data is generated |
| Classification | Uses Bayes' rule: $p(y\|x) \propto p(x\|y)p(y)$ | Directly estimates $p(y\|x)$ or a decision function |
| Flexibility vs. Complexity | Can be more flexible (e.g., handling missing data) but may need more data to estimate the full joint distribution | Often simpler for classification and may generalize better for prediction tasks |

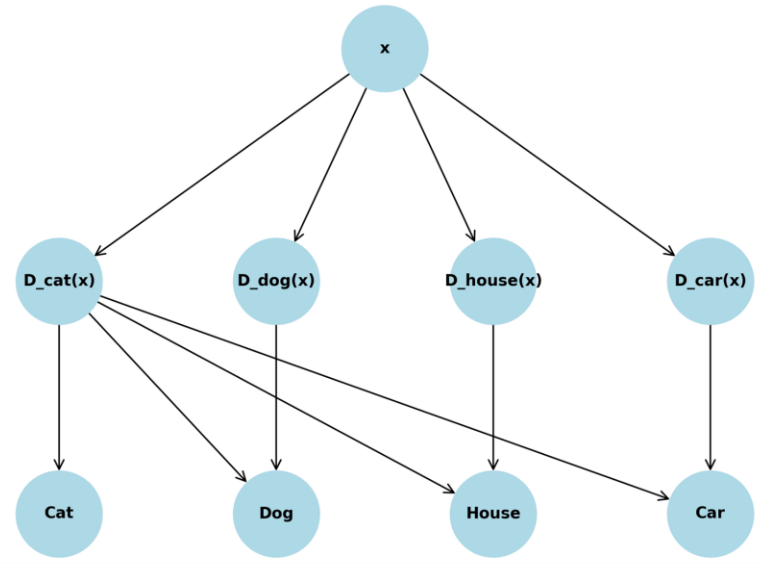
# Formal representation of *D*

$Y = \{y_1, y_2, \ldots, y_k\}$

$D_i(X) = P(y_i|X)$

$D_i(X) > D_j(X)$, for all j != i => predict $y_i$



# Parameter Estimation Methods

These are different methods used to understand how models learn from data. In other words, how to learn model parameters given the input data.

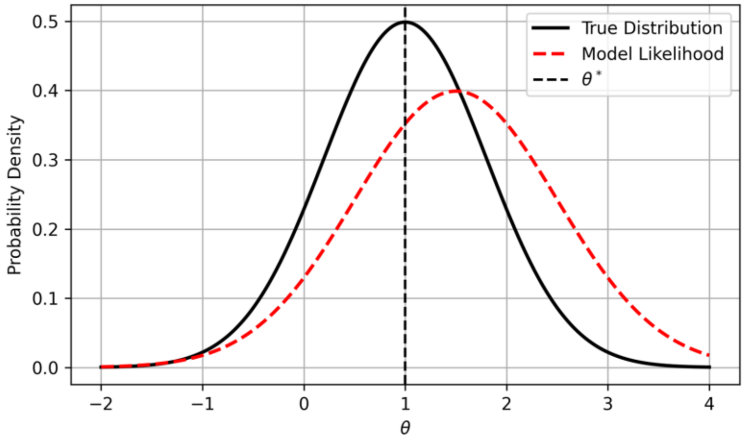
| Criterion | MLE | MAP | LS | MoM | Bayesian | EM |
|---|---|---|---|---|---|---|
| Objective | Maximize $p(x\|\theta)$ | Maximize $p(\theta\|x)$ | Minimize squared error | Match sample and theoretical moments | Compute full posterior $p(\theta\|x)$ | Maximize expected complete log-likelihood |
| Incorporation of Prior | No | Yes | No | No | Yes | (Yes, via MAP-EM variant) |
| Computational Complexity | Moderate to high | Similar to MLE + prior handling | Low (for linear models) | Low | Often high (integration or sampling) | Moderate, iterative |
| Applicability | Well-specified likelihood models | When prior knowledge is available | Regression and related tasks | When moments are easily computed | Full uncertainty quantification needed | Models with latent variables/missing data |
| Concerns | Sensitive to model misspecification | Dependent on chosen prior | Sensitive to outliers | May be biased in small samples | Computationally intensive | May converge to local optima |
| Examples | Logistic Regression, HMM (via MLE) | Regularized Regression, MAP estimates in Bayesian networks | Linear Regression | Moment matching for Gaussian or Beta distributions | Bayesian Linear Regression, MCMC, Variational Inference | Gaussian Mixture Models, Hidden Markov Models |

# MLE - Maximum Likelihood Estimation

x = $\{x_1, x_2, \ldots, x_n\}$ are i.i.ds from true distribution Px(x)

We want to learn a model with parameter θ such that the difference between Px(x) and P(x|θ) is minimized.

P(x|θ) => likelihood of the model




Given a training dataset $\mathcal{D} = (x_i, y_i)_{i=1}^{n}$ of $n$ independent and identically distributed samples from the true distribution $P(X, Y)$, the likelihood function for a discriminative model is:

$$L(\theta \mid \mathcal{D}) = \prod_{i=1}^{n} P\left(y_i \mid x_i; \theta\right)$$

Taking the logarithm for computational convenience:

$$\log L(\theta \mid \mathcal{D}) = \sum_{i=1}^{n} \log P\left(y_i \mid x_i; \theta\right)$$

The maximum likelihood estimator is then:

$$\hat{\theta}_{\mathrm{MLE}} = \arg\max_{\theta} \sum_{i=1}^{n} \log P\left(y_i \mid x_i; \theta\right)$$

# MAP - Example 1

$$X_1, \ldots, X_n \sim \text{i. i. d Bernoulli}(p)$$

- Samples: $x_1, x_2, \ldots, x_n$
  $x_i = 0$ or 1
- Likelihood: $L(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$
  - $f_X(x_i) = p$ if $x_i = 1$, or $f_X(x_i) = 1 - p$ if $x_i = 0$
  - Let $w$ denote the number of 1 s in the sample

$$L(x_1, \ldots, x_n) = p^w (1-p)^{n-w}$$

Maximise function: "Differentiate and equate to zero"

$$\rightarrow \text{ Maximizing } L \leftrightarrow \text{ Maximizing } \log L$$

$$h(p) = \log L = w \log p + (n - \omega) \log(1-p)$$

$$\frac{dh(p)}{dp} = \omega \times 1/p + (n - w) \frac{1}{1-p}(-1) = 0$$

$$w(1-p) = (n - \omega)p$$

$$p = \omega / n$$

# MAP - Maximum A Posteriori Estimate

MAP estimation incorporates prior knowledge about parameters through Bayes' rule:

$$P(\theta|X,Y) \propto P(Y|X,\theta) \cdot P(\theta)$$

Till now, $P(X \mid \theta) \leftarrow$ any likelihood. [not included prior yet — non-bayesian]

$$P(\theta \mid x) = \frac{P(x \mid \theta) P(\theta)}{P(x)}$$

$$L_{\text{map}} = \log P(\theta \mid x) = \log P(x \mid \theta) + \log P(\theta) + C$$

$$\hat{\theta}_{\text{map}} = \underset{\theta}{\operatorname{argmax}}\, L_{\text{MLE}}(\theta) + \log P(\theta)$$

# Linear : MLE :: Ridge : MAP

$$y = \omega^\top x + \epsilon \sim N\left(0, \sigma^2\right)$$

for $MLE$,

$$\begin{aligned} P(y \mid x, \omega) &= N\left(y; \omega x, \sigma^2\right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} 2^{-(y-\omega^\top x)^2/2\sigma^2} \end{aligned}$$

for MAP,

$$\begin{aligned} P(\omega \mid x, y) = \frac{P(x, y, \omega)}{P(x, y)} &\propto P(y \mid x, \omega) P(x, \omega) \\ &\propto P(y \mid x, \omega) P(\omega) P(x \mid \omega) \\ \log p(\omega \mid x, y) &= \log P(y \mid x, \omega) + \log P(\omega) \end{aligned}$$

Given, $\quad P(\omega) = N\left(\omega; 0, \frac{1}{\lambda} I\right) \rightarrow$ (uniform prior, condition for linear regression)

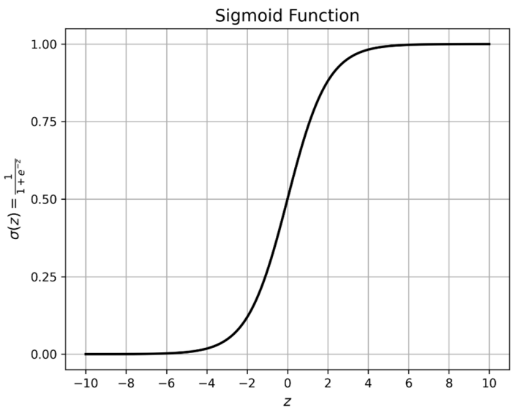
$$\omega^* = \operatorname*{argmax}_\omega \log p(y \mid x, \omega) + \log p(\omega)$$

$$= \operatorname*{argmax}_\omega \log \frac{1}{\sigma\sqrt{2\pi}} + \log\left(e^{-\left(y-\omega^\top x\right)^2/2\sigma^2}\right) + \log\left(e^{-(\omega-0)^\top \Sigma^{-1} \omega}\right)$$

$$\begin{aligned} L(\omega) &= -\frac{y - \left(\omega^\top x\right)^2}{2\sigma^2} - \lambda\|\omega\|^2 \\ \omega^* &= \operatorname{argmax} L(\omega) \qquad \text{Let, } \sigma = 1 \\ &= \arg\min_\omega \frac{\left(y - \omega^\top x\right)^2}{2} + \lambda\|\omega\|^2 \end{aligned}$$

# Logistic Regression: **Intro**

- Supervised algorithm for classification
- Based on “odds ratio”
- Odds = P / (1-P) => that is why sigmoid is used




Data: $\{(x_i, y_i)\}_{i=1}^{n}, \quad x_i \in \mathbb{R}^d$

$$P\left(y_i = 1 \mid x_i, \omega\right) = \text{Benn}\left(\sigma\left(\omega^\top x\right)\right) = P_i$$

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

MLE: $\theta^* = \underset{\theta}{\operatorname{argmax}} \underset{x,y}{E}[\log(P(x,y) \mid \theta)]$

$$= \underset{\theta}{\operatorname{argmax}} \underset{x,y}{E}\left[\log \frac{P(x,y,\theta)P(\theta,x)}{P(\theta)P(\theta,x)}\right]$$

$$= \underset{\theta}{\operatorname{argmax}} \underset{x,y}{E}[\log P(y \mid x,\theta), P(x \mid \theta)]$$

$$\theta^* = \underset{\theta}{\operatorname{argmax}} \underset{x,y}{E}[\log p(y \mid x,\theta)]$$

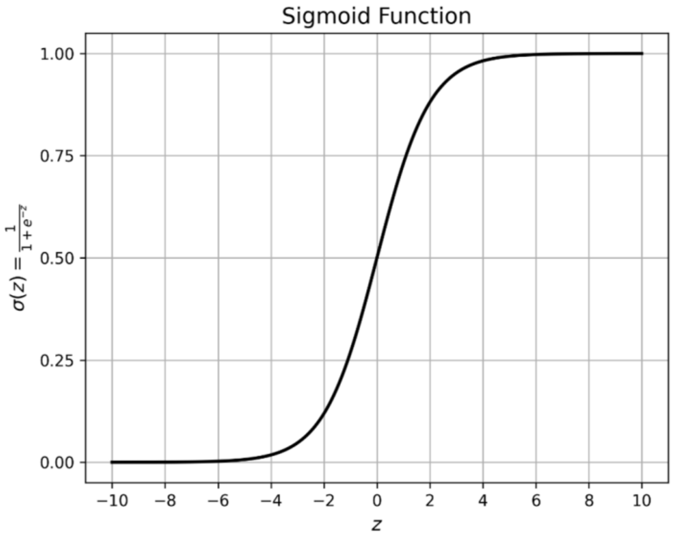
$$\theta^* = \underset{\theta}{\operatorname{argmax}} \frac{1}{N} \sum_{i=1}^{n} \log P\left(y_i \mid x_i, \theta\right)$$

# Logistic Regression: **MLE**

$$\mathrm{Bern}\left(y_i; p_i\right) = p_i^{y_i}\left(1 - p_i\right)^{1-y_i}$$

For logistic,

$$\begin{aligned}\omega^* &= \underset{\omega}{\operatorname{argmax}} \frac{1}{N}\sum_{i=1}^{N} \log P_i^{y_i}\left(1 - P_i\right)^{1-y_i} \\ &= \underset{\omega}{\operatorname{argmin}} - \frac{1}{N}\sum_{i=1}^{N} \left\{y_i \log P_i + \left(1 - y_i\right)\log\left(1 - P_i\right)\right\}\end{aligned}$$



# Logistic Regression: **Gradient Descent**

$$\omega^{(t+1)} = \omega^{(t)} - \eta_t \nabla_\omega L\left(\omega^{(t)}\right)$$

$$P_i = \sigma\left(z_i\right)$$

$$z_i = \omega^\top x_i$$

$$\frac{\partial L}{\partial \omega^j} = \frac{\partial L_i}{\partial P_i} \cdot \frac{\partial P_i}{\partial z_i} \cdot \frac{\partial z_i}{\partial \omega^j}$$

$$\frac{\partial L_i}{\partial P_i} = \frac{-1}{N}\left[\frac{y_i}{P_i} - \frac{1-y_i}{1-P_i}\right]$$

$$\frac{\partial P}{\partial z} = \sigma'(z) = \sigma(z)(1-\sigma(z))$$

$$\frac{\partial z_i}{\partial \omega^j} = \frac{\partial}{\partial \omega^j}\left[\omega_0 + \omega^1 x_i^1 + \omega^2 x_2^2 + \cdots\right]$$

$$= x_i^j$$

$$\frac{\partial L}{\partial \omega^j} = (1) \times (2) \times (3)$$

$$= -\frac{1}{N}\sum_{i=1}^{N}\left[\frac{y_i}{P_i} - \frac{1-y_i}{1-P_i}\right] P_i\left(1-P_i\right) x_i^j$$

# Logistic Regression: **Gradient Descent**

$$= -\frac{1}{N}\sum_{i=1}^{N}\left[y_i\left(1-p_i\right)-\left(1-y_i\right)p_i\right]x_i^j$$

$$\frac{\partial Li}{\partial \omega^j} = -\frac{1}{N}\sum_{i=1}^{N}\left(y_i - p_i\right)x_i^j$$

$$\nabla_\omega L\left(\omega^{(t)}\right) = \begin{bmatrix} \frac{\partial L}{\partial \omega^0} \\ \frac{\partial L}{\partial \omega'} \\ \vdots \\ \frac{\partial L}{\partial \omega^\alpha} \end{bmatrix} = -\frac{1}{N}x^\top\left(y-\sigma(x\omega)\right)$$

G.D: $\quad \omega^{(t+1)} = \omega^{(t)} - \eta\frac{1}{N}x^\top\left(\sigma(\omega)-y\right)$

Inferenc:?

# Logistic Regression: **Multiclass**

$$C = \{C_1, L_2, \dots C_k\}$$

$$C_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\sum p_i = 1, \quad p_i > 0$$

$$p_i \propto e^{\omega_i^\top x}$$

$$\Rightarrow P_i = \frac{e^{\omega_i^\top x}}{\sum_{j=1}^k e^{\omega_j^\top x}}$$

$$P\left(y = c_i \mid x_i, \omega\right) = \text{ multinomial } \left\{ y_i; \frac{e^{\omega_1^\top x}}{s}, \frac{e^{\omega_2^\top x}}{S}, \cdots, \frac{e^{\omega_k^\top x}}{S} \right\}$$

$$\omega^* = \operatorname*{argmin}_\omega \mathop{E}_{x,y} [-\log(p(y \mid x, \omega))]$$

$$= \operatorname*{argmin}_\omega \sum_{i=1}^{w} \sum_{j=1}^{k} y_i^j \log p_i^j$$

$$\nabla L = -\frac{1}{N} x^\top (y - \sigma(x\omega))$$

# Logistic Regression: **Code Walk through**

# Decision Tree

| Match | Pitch Type | Host | Batting First | Winner |
|---|---|---|---|---|
| M1 | Spin-friendly | India | India | India |
| M2 | Pace-friendly | Australia | Australia | Australia |
| M3 | Balanced | India | Australia | India |
| M4 | Spin-friendly | Australia | India | Australia |
| M5 | Pace-friendly | India | Australia | Australia |
| M6 | Spin-friendly | India | Australia | India |
| M7 | Balanced | Australia | India | India |
| M8 | Pace-friendly | Australia | India | Australia |
| M9 | Spin-friendly | India | India | India |
| M10 | Balanced | Australia | Australia | Australia |

# Decision Tree

$$P(x=1) = P$$

$$P(x=0) = 1 - P$$

$$H(x) = \sum_{z=1}^{k} -P(x=i)\log P(x=i)$$

$$= -\int_{-\infty}^{\infty} P(x)\log P(x)dx$$

$$= E_{x\sim P(x)}[-\log P(x)]$$

$$H(\mathrm{Bern}\, n(P)) = -P\log P - (1-P)\log(1-P)$$

$$\frac{\partial H}{\partial P} = \frac{-P}{P} - \log P + \log(1-P)$$

$$H = 0$$

$$\log\left(\frac{1-P)}{P}\right) = 0$$

$$\Rightarrow 1 - P = P$$

$$\Rightarrow P = 1/2$$