Lecture 26: An example illustrating the LP duality based Shortest Path algorithm

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1 Formulations for an example graph

Let us see the formulation for an example graph in Figure 1.

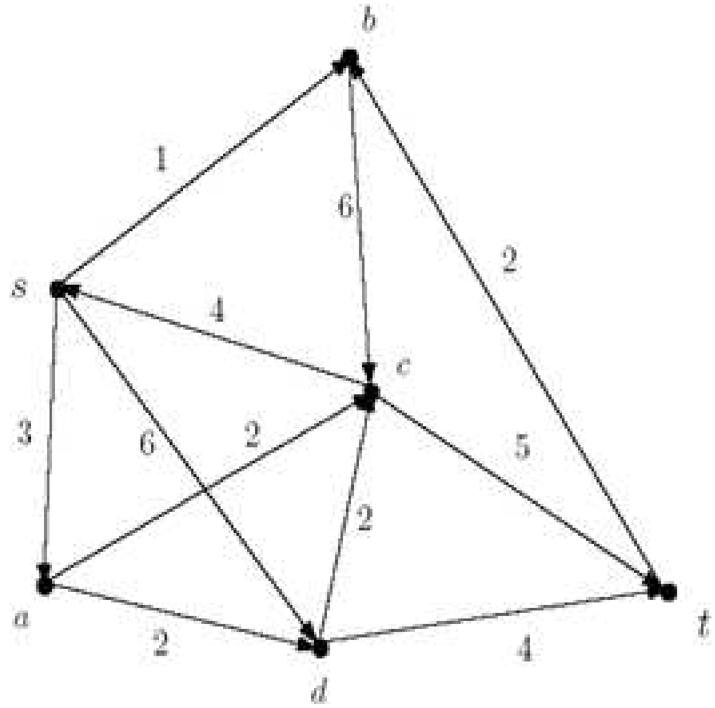

Figure 1: Given input graph

The integer LP is:

$$\min x_{sb} + 3 * x_{sa} + 6 * x_{sd} + 2 * x_{ad} + 2 * x_{ac} + 6 * x_{bd} + 4 * x_{cs} + 5 * x_{ct} + 4 * x_{dt} + 2 * x_{tb} \quad (1)$$

$$x_{sa} + x_{sb} + x_{sd} = 1 \quad (2)$$

$$x_{ct} + x_{dt} = 1 \quad (3)$$

$$\forall v \in V - \{s,t\} \sum_{w} x_{vw} - \sum_{u} x_{uv} = 0 \quad (4)$$

$$x_{uv} \geq 0 \tag{5}$$

$$x_{uv} \text{ is an integer} \tag{6}$$

To obtain the linear program, we simple drop the constrant that x_{uv} are integers.

And the corresponding dual is

$$\max\ y_s - y_t \tag{7}$$

$$\forall u, v \qquad y_v - y_u \leq w_{uv} \tag{8}$$

$$y_s - y_a \leq 3 \tag{9}$$

$$y_s - y_b \leq 1 \tag{10}$$

$$y_s - y_d \leq 6 \tag{11}$$

$$y_a - y_c \leq 2 \tag{12}$$

$$y_a - y_d \leq 2 \tag{13}$$

$$y_b - y_c \leq 6 \tag{14}$$

$$y_c - y_s \leq 4 \tag{15}$$

$$y_c - y_t \leq 5 \tag{16}$$

$$y_d - y_t \leq 4 \tag{17}$$

Initially we set $y_s = y_a = y_b = y_c = y_d = y_t = 0$. Now we keep incrementing y_i's till some $y_v - y_u \leq w_{uv}$ becomes equality keeping $y_t = 0$.

Iteration Number	y_s	y_a	y_b	y_c	y_d	y_t	Remark
1	1	1	1	1	1	0	
2	2	2	2	2	2	0	
3	3	3	3	3	3	0	
4	4	4	4	4	$\underline{4}$	0	$y_d - y_t = w_{dt} = 4$. Hence $y_d = 4$
5	5	5	5	$\underline{5}$	$\underline{4}$	0	$y_c - y_t = w_{ct} = 5$. Hence $y_c = 5$
6	6	$\underline{6}$	6	$\underline{5}$	$\underline{4}$	0	$y_a - y_d = w_{ad} = 2$. Hence $y_a = 6$
7	7	$\underline{6}$	7	$\underline{5}$	$\underline{4}$	0	
8	8	$\underline{6}$	8	$\underline{5}$	$\underline{4}$	0	
9	$\underline{9}$	$\underline{6}$	9	$\underline{5}$	$\underline{4}$	0	$y_s - y_a = w_{sa} = 3$. Hence $y_s = 9$. Stop now.

Table 1: Trace of the primal-dual algorithm on the example graph

See Figure 2 for intermediate steps. Graph obtained after these iteration is 2(v). It contains some extra edges. So we apply reverse delete algorithm. First we delete maximum weight edge e_{ct} and check for a path from s to t. Clearly such a path $s \Rightarrow a \Rightarrow d \Rightarrow t$ exist. Next we try to remove e_{dt}. This makes t unreachable from s. Hence we stop here. Graph thus obtained is shown in Figure 2(vi). Shortest path thus obtained is $s \Rightarrow a \Rightarrow d \Rightarrow t$.

We also see that value of shortest path thus obtained is $y_s - y_t = 9$.

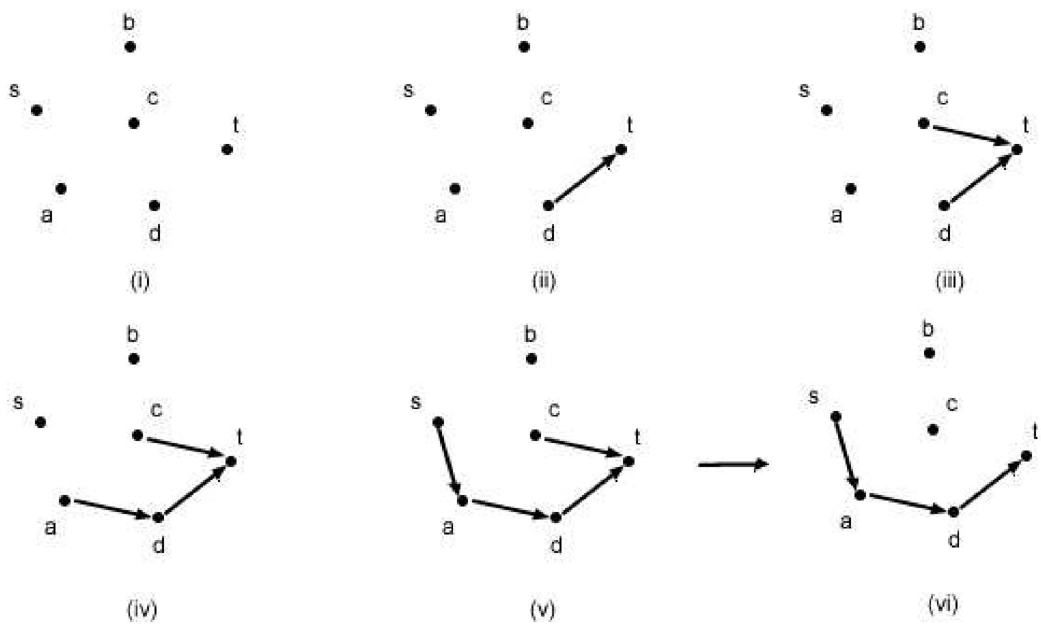

Figure 2: Intermediate Steps