

**Lecture 1: Introduction**Lecturer: *Sundar Vishwanathan*Scribe: *Anup Kumar*

COMPUTER SCIENCE &amp; ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

## 1 History of Linear Optimization

This subject first came up in manufacturing and industrial engineering in 1940s where they need to derive maximum profit with less expenditure.

## 2 Definition of Linear Programming

We illustrate the problem of Linear Programming through the following example.

**EXAMPLE 1** A factory needs to produce different goods from different kinds of raw materials available. Each good yields a certain amount of profit per unit. Each good in the process of its manufacture uses a certain proportion of each available raw material. The amount of each raw material available is limited.

The problem is how much quantity of each good to produce using these raw materials in order to maximize the total profit on the goods.

The problem is mathematically formulated as follows :

- Raw Materials Available:  $M_1, M_2, M_3, \dots, M_m$
- Quantity of each material available:  $b_1, b_2, b_3, \dots, b_m$
- Goods to be produced:  $G_1, G_2, G_3, \dots, G_n$
- Profit on each good:  $p_1, p_2, p_3, \dots, p_n$
- $A_{ij}$ : Proportion of material  $M_i$  required to produce one unit quantity of good  $G_j$
- $x_j$ : Quantity of each good  $G_j$  produced.  $x_j \geq 0$ .
- Total Profit: The sum of profit on all goods =  $\sum_j x_j p_j$

**Optimization Problem:** How much of each good to be produced,  $x_j$ , to maximize the total profit under the given constraints?

The name *Linear Optimization* comes from the fact that the quantity which is to be optimized is a linear function of the unknown quantity  $x_j$  and the constraints on  $x_j$  are linear inequalities. The above problem is an instance of maximizing the value.

There can be problem instances where we have to minimize some quantity and find out the unknowns which minimize that quantity. For example, the above problem could have been formulated as minimum raw material requirement under certain constraints such as given profit, etc.

**Problem Statement:**

- Inputs:  $b_i$ ,  $p_j$ , the matrix  $A$
- Outputs:  $x_j$ , which denotes the quantity of each  $G_j$  to be produced
- To Maximize:  $\sum_j x_j p_j$ . If profit  $p_j$  is denoted by an  $n \times 1$  column matrix  $p$ , then we have to maximize the value  $p^t x$  where  $x$  denotes the column matrix of  $x_j$ s.
- Constraints:
  - for all material  $i$ ,  $\sum_j a_{ij} x_j \leq b_i$  or  $Ax \leq b$
  - $x_j \geq 0$

**3 Solution Idea**

We need to understand what  $\{x : Ax \leq b\}$  looks like. Suppose there were only two variables  $x_1$  and  $x_2$ . We begin by considering the object  $\{x : Ax = b\}$ . The linear equations  $\sum_j a_{ij} x_j = b_i$  represents a set of straight lines in two dimensional plane with  $x_1$  and  $x_2$  as co-ordinates.

The solution to these equations can be a single point in the plane or a line or empty. This can be extended further when there are 3 or more variables. Our goal is to have a geometric interpretation and an algebraic description for these sets.

In the next class we shall see the solutions to the linear equations and their geometric description.