

Lecture 10: Extreme Points

Lecturer: *Sundar Vishwanathan*Scribe: *Swati Goyal*

COMPUTER SCIENCE & ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

1 Extreme Points

DEFINITION 1 *An extreme point in a convex set is a point which cannot be represented as a convex combination of two other points of the set.*

In the last lecture, we proved that in a space of dimension n , an extreme point is the intersection point of n linearly independent hyperplanes i.e

Take an x_0 such that $Ax_0 \leq b$ given by

$$A'x_0 = b' \tag{1}$$

$$A''x_0 < b'' \tag{2}$$

then x_0 is an extreme point iff A' consists of n linearly independent rows(hyperplanes). Note that we have assumed $Ax_0 \leq b$ to be non-degenerate.

2 Convex Combination of Extreme Points

Consider a region $Ax \leq b$

Let p_1, p_2, \dots, p_n be the extreme points of this region. Then this region can also be represented as the convex hull of these extreme points.

We will prove this statement in the following section.

THEOREM 1

Any point x , such that $Ax \leq b$, can be written as convex combination of the extreme points of this region.

PROOF: The proof will use induction on dimension of the region.

Base Case: Dimension of the region is 0. Then there is just one point and hence the theorem is vacuously true. We illustrate the main technique by looking at an example in 2 dimension.

Consider the region in Fig1.



Figure 1: Convex polygon in dimension 2

Let p be any point in the given region and say $p_1, p_2 \dots$ be the extreme points of the region. Now join p_1 and p and extend it to intersect p_3p_4 at q . Since q lies on the line joining p_3p_4 it can be written as a convex combination of p_3 and p_4 .

$$q = \Sigma \alpha_i p_i, 0 \leq \alpha_i \leq 1, \Sigma \alpha_i = 1, i = 3, 4 \quad (3)$$

$$p = \beta_1 p_1 + \beta_2 q, 0 \leq \beta_i \leq 1, \Sigma \beta_i = 1 \quad (4)$$

and hence

$$p = \Sigma \gamma_i p_i, 0 \leq \gamma_i \leq 1, \Sigma \gamma_i = 1 \quad (5)$$

Hypothesis: Assume that any x , such that $Ax \leq b$ in space of dimension $n - 1$ can be written as convex combination of the extreme points of the region.

Induction: Consider a region $S : Ax \leq b$ in space of dimension n with $p_1, p_2 \dots$ as the extreme points

Consider a point $p \in S$. Join p_1 and p and extend the line segment to meet a point q on the boundary of S . At this boundary, at least one of the inequalities in $Ax \leq b$ has to be equality. For simplicity assume that the first is equality.

Thus at q ,

$$A_1 q = b_1 \quad (6)$$

$$A'' q < b'' \quad (7)$$

where $q = [q_1, q_2, \dots]$

Solve (6) for some q_i and substitute its value in (7).

Thus (7) now reduces to another region S' of dimension $n - 1$ of the form $Cx \leq D$.

By assumed hypothesis, q can be written as convex combination of extreme points of S' .

$$q = \Sigma \alpha_i (p')_i \text{ where } \Sigma \alpha_i = 1 \text{ and } 0 \leq \alpha_i \leq 1 \quad (8)$$

Now p can be written as a convex combination of q and p_1 i.e. as a convex combination of $p'_1 \dots$ and p . But this still doesn't prove that p is a convex combination of $p_i \dots$ (extreme points of S)

For this we would have to prove that the extreme points of S' are a subset of the extreme points of S .

Assume this is not true, that is, p' is an extreme point of S' but not of S .

Since, $p' \in S'$ and as S' is a subset of S , thus $p' \in S$.

Also, $\exists p_1, p_2 \in S$ such that p' is the convex combination of p_1, p_2 (as p' is not an extreme point in S).

$$p' = \lambda p_1 + (1 - \lambda)p_2 \quad 0 \leq \lambda \leq 1 \quad (9)$$

Now, for p' to be an extreme point in S ,

$$A_1 p' = b_1 \quad (10)$$

must be true, from (6), as (7) is already satisfied for all points in region S' .

i.e.

$$A_1 \lambda p_1 + A_1 (1 - \lambda)p_2 = b_1 \quad \text{from(9)and(10)} \quad (11)$$

Since $A_1 p_1 \leq b_1$ and $A_1 p_2 \leq b_1$, both of them have to be equal to b_1 . Thus, p_1 and p_2 satisfy (6) and thus belong to S' .

This leads to a contradiction since p' (extreme point) cannot be expressed as convex combination of two or more points of the region.

Hence, the extreme points of S' are a subset of the extreme points of S .

Thus,

$$q = \sum \alpha_i p'_i = \sum \alpha_i p_i \quad (12)$$

$$p = \beta_1 p_1 + \beta_2 q \quad (13)$$

$$p = \sum \gamma_i p_i \quad \text{from(12)and(13)} \quad (14)$$

Thus, every point in S can be expressed as the convex combination of the extreme points of S . \square

3 Maximize $c^T x$

Given a linear function $c^T x$ and a region $Ax \leq b$, such that the dimension of the space is n and the number of rows in A is m , then the number of the extreme points of the region is $C(m, n)$. Each extreme point is an intersection of n linearly independent planes, and thus the number of extreme points is at most the number of possible ways of choosing n planes from m choices.

But before finding the maximum value for the function in the given region, we will have to prove a theorem.

THEOREM 2

The maximum value of a linear function $f(x)$ occurs at an extreme point of the region $Ax \leq b$.

PROOF: Let f be maximum at point p of the region $Ax \leq b$.

From the previous theorem we have,

$$p = \sum \alpha_i p_i \quad 0 \leq \alpha_i \leq 1 \quad \sum \alpha_i = 1 \quad (15)$$

where $p_1 \dots$ are the extreme points of S.

Thus,

$$f(p) = f\left(\sum \alpha_i p_i\right) \quad (16)$$

$$f(p) = \sum \alpha_i f(p_i) \quad (17)$$

since f is a linear function.

$$\sum \alpha_i f(p_i) \leq \theta \sum \alpha_i \quad (18)$$

where θ is the maximum value among $f(p_i)$. Thus

$$\theta = f(p) = \sum \alpha_i f(p_i) \leq \theta \sum \alpha_i \leq \theta \quad (19)$$

Thus there must exist some extreme point p_i such that $f(p) = f(p_i)$, otherwise the equality cannot be satisfied in (19).

Hence, the maximum value of f occurs at an extreme point. \square

In next lecture, we will look at the simplex algorithm for finding the extreme points of the region $Ax \leq b$.