Lecture 12: The Simplex Algorithm III

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In this lecture we will give the stopping condition of the Simplex algorithm and also prove that the algorithm is correct.

To recap, at a given extreme point x_0:

1. There are matrices $A^{'}$, $b^{'}$, $A^{''}$ and $b^{''}$ (constructed from A and b) s.t. $A^{'}x_0 = b^{'}$ and $A^{''}x_0 \leq b^{''}$.

2. The directions of the neighbouring extreme points are the columns of the matrix $-A^{'^{-1}}$.

Stopping Condition *The algorithm stops at an extreme point x_0 and returns it as optimal when the cost at all the neighbouring extreme points of x_0 is less than that at x_0.*

1 Proof of Correctness

We now try to prove that the Simplex algorithm is correct i.e. when it terminates, we indeed have found the globally optimal point.

First of all note that this is *not* the same as saying that x_0 is local maximum and hence by a previously proved theorem, it is also a global maximum. This is so because we just know that the cost at x_0 is maximum as compared to its *neighbours* - not compared to a small enough *neighbourhood* around it.

1.1 First Approach

One approach would be to consider a small enough neighbourhood N around x_0. Now suppose we somehow prove that any point $p \in N$ can be written as a convex combination of all the neighbours of x_0, i.e.:

$$p = \sum_{i=0}^{n} \lambda_i x_i \tag{1}$$

$$\sum \lambda_i = 1 \tag{2}$$

Then we are clearly done because we know that $\forall i \; c^T x_i \leqslant c^T x_0$. Hence,

$$c^T p = \sum \lambda_i c^T x_i \leqslant c^T x_0 (\sum \lambda_i) = c^T x_0 \tag{3}$$

Thus, x_0 is a local maximum and consequently a global maximum.

1.2 Second Approach

We adopt a different approach than the above for the proof. Assume that x_0, where the algorithm terminates, is not an optimal point. Also, suppose that there is some other optimal point x_{opt}. Therefore, $c^T x_{opt} > c^T x_0$.

Now as x_0 is an extreme point, $A^{'}$ has full rank. So, even $-A^{'^{-1}}$ has full rank. Or in other words, the n columns form a basis of the space. Hence, the vector $x_{opt} - x_0$ can be written as a linear combination of these columns, i.e.:

$$x_{opt} - x_0 = \sum_i \beta_i (-A^{'^{-1}})^{(i)} \tag{4}$$

where $B^{(i)}$ represents the i^{th} column of a matrix B. Pre-multiplying with $A^{'}$ in the above equation, we get

$$A^{'} x_{opt} - A^{'} x_0 = \sum_i \beta_i A^{'} (-A^{'^{-1}})^{(i)} \tag{5}$$

Note the following in the above equation:

1. As x_{opt} is a feasible point, $A^{'} x_{opt} \leqslant b^{'}$ whereas $A^{'} x_0 = b^{'}$. Hence $A^{'} x_{opt} - A^{'} x_0$ will be a vector with each component ≤ 0.
2. $A^{'} (-A^{'^{-1}})^{(i)} \leq \mathbf{0}$ or to be more specific it has a zero at all positions except at the i^{th} row where it is -1.

These two observations imply that $\forall j \; \beta_j \geqslant 0$.

Now pre-multiply with c^T in Equation 4. We get

$$c^T x_{opt} - c^T x_0 = \sum_i \beta_i c^T (-A^{'^{-1}})^{(i)} \tag{6}$$

We know that as $(-A^{'^{-1}})^{(i)}$ are directions of the neighbours, $(-A^{'^{-1}})^{(i)} = \alpha_i (x_i - x_0)$ where x_i's are the neighbours of x_0 and $\alpha_i \geq 0$. As we have stopped at x_0, $\forall x_i \; c^T x_i - c^T x_0 \leq 0$. Coupled with the fact that $\beta_j \geqslant 0$, the R.H.S. of the above equation is ≤ 0. Hence, we get

$$c^T x_{opt} - c^T x_0 \leq 0 \tag{7}$$

Comparing this with our assumption, we find that there is a contradiction. So, our assumption is wrong and x_0 is indeed an optimal point.