Lecture 12: The Simplex Algorithm III

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Let x_0 be the current extreme point. From previous lectures we already know that:

- The directions of the neighboring extreme points of x_0 are given by the columns of the matrix $-A'^{-1}$
- We can write $Ax_0 \leq b$ as
$$A'x_0 = b' \tag{1}$$
$$A''x_0 < b'' \tag{2}$$
where A' consists of n linearly independent rows(hyperplanes).

In this lecture we will give the stopping condition of the Simplex algorithm and prove the correctness of the algorithm.

1 Stopping Condition

If all the neighboring extreme points of x_0 have a cost $\leq$ the cost of x_0 then x_0 is optimal and the algorithm stops at x_0.

Proof First of all note that this is *not* the same as saying that x_0 is local maximum and hence by a previously proved theorem, it is also a global maximum. This is so because we just know that the cost at x_0 is maximum as compared to its *neighbours* - not compared to a small enough *neighbourhood* around it.

First Approach

Let us consider a small neighborhood N of x_0. Also, assume that we can write any point $p \in N$ as a convex combination of all the neighboring extreme points of x_0 i.e.

$$p = \sum_{i=0}^{n} \lambda_i x_i \quad ; \quad 0 \leq \lambda_i \leq 1, \Sigma\lambda_i = 1 \tag{3}$$

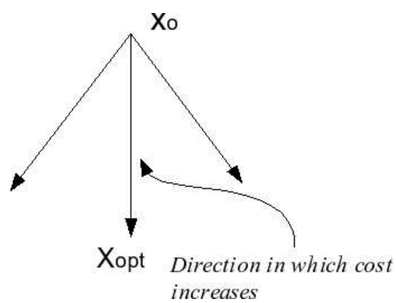
This is similar to taking a weighted average with $\lambda_i s$ be the probabilities.
Now we know that $\forall i \; c^T x_i \leq c^T x_0$. Hence,

$$c^T p = c^T (\sum_{i=0}^{n} \lambda_i x_i) = \sum_{i=0}^{n} \lambda_i c^T x_i \leq c^T x_0 (\sum_{i=0}^{n} \lambda_i) = c^T x_0 \quad (4)$$

Thus, x_0 is a local maximum and consequently a global maximum.
To be shown : (i) $p = \sum \lambda_i x_i$ (ii) Starting criteria

Second Approach

Assume that x_0 is not an optimal point and there is some other point x_{opt} which is optimal. Thus, the cost increases along $x_{opt} - x_0$.



Now, x_0 is an extreme point

$\Rightarrow A^{'}$ has full rank
$\Rightarrow -A^{'^{-1}}$ has full rank
$\Rightarrow n$ columns form a basis of the space
$\Rightarrow$ vector $x_{opt} - x_0$ can be written as a linear combination of these columns. Hence,

$$x_{opt} - x_0 = \sum_i \beta_i ((-A^{'^{-1}})^i) \quad (5)$$

Pre-multiplying with $A^{'}$ in the above equation, we get

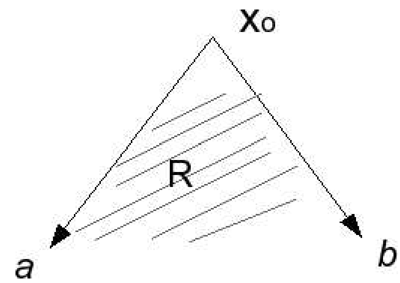
$$A^{'} x_{opt} - A^{'} x_0 = \sum_i \beta_i A^{'} ((-A^{'^{-1}})^i) \quad (6)$$

What can we say about β_i ?

Lets look at the following figure:

Any point in the region R can be written as $a\beta_1 + b\beta_2$. For feasibility, β_1 and β_2 have to be positive, otherwise we will be out of the region R ($Ax \leq b$). Thus, intutively it should be clear that $\beta_i \geqslant 0$.

More formal proof for $\beta_i \geqslant 0$ will be given in the next lecture.



Now pre-multiplying with c^T in (5), we get

$$c^T x_{opt} - c^T x_0 = \sum_i \beta_i c^T((-A^{'-1})^i) \tag{7}$$

Now, $(-A^{'-1})^i = x_i - x_0$ where x_i's are the neighboring extreme points of x_0. Since we have stopped at x_0, $\forall x_i \; c^T x_i - c^T x_0 \leq 0$. Also, $\beta_j \geqslant 0$. This implies that the R.H.S. of the above equation is $\leqslant 0$. Hence,

$$c^T x_{opt} - c^T x_0 \leqslant 0 \tag{8}$$

Thus, we find that cost decreases along $x_{opt} - x_0$ which is a contradiction. So, our assumption was wrong and x_0 is indeed an optimal point. Hence proved !