Lecture 13: Proof of correctness of Simplex Algorithm(contd) and Introduction to Duality Theorem

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Let x_0 be an extreme point. Suppose it is given by :

$$A^{'}x_0 = b^{'}, A^{''}x_0 < b^{''} \tag{1}$$

From lecture 11 we know that the neighbours of x_0 are along the columns of $-A^{'-1}$

1 Proof of correctness of Simplex algorithm

Theorem 1 *If the cost decreases along the columns of $-A^{'-1}$ then x_0 is optimal.*

Proof: The columns of $-A^{'-1}$ span R^n. Let x_{opt} be an optimal point i.e. $c^T x_{opt} \geq c^T x_0$ then we need to show that $c^T x_{opt} \leq c^T x_0$ to establish $c^T x_{opt} = c^T x_0$ and hence $x_{opt} = x_0$. Since the columns of $-A^{'-1}$ are a basis the vector $x_{opt} - x_0$ can be represented as a linear combination of them.

$$x_{opt} - x_0 = \sum \beta_j (-A^{'-1})^j \tag{2}$$

Now consider $A'(x_{opt} - x_0)$

$$A'x_{opt} - A'x_0 = \sum \beta_j A'(-A^{'-1})^j \tag{3}$$

We know that $A'x_{opt} \leq b' and A'x_0 = b'$ hence $A'(x_{opt} - x_0) \leq 0$. Also note that $A'(-A^{'-1})^j$ is an $n \times 1$ vector whose j^{th} element is -1 and remaining elements are 0. Hence

$$A'x_{opt} - A'x_0 = \begin{pmatrix} -\beta_1 \\ -\beta 2 \\ . \\ . \\ . \\ -\beta n \end{pmatrix} \tag{4}$$

$\Rightarrow \forall j \beta_j >= 0$
From the discussion above we infer that $\beta_j \geq 0$ for each j.
Now consider

$$c^T x_{opt} - c^T x_0 = \sum \beta_j c^T (-A^{'-1})^j \tag{5}$$

Since the cost decreases along the columns of $-A^{'-1}$ we have $c^T(-A^{'-1})^j \leq 0$ and since $\beta_j \geq 0$ we conclude that $\sum \beta_j c^T (-A^{'-1})^j \leq 0$
Hence $c^T x_{opt} \leq c^T x_0$ but we know that $c^T x_{opt} \geq c^T x_0$ and $c^T x_{opt} = c^T x_0$. □

Note: Using the above theorem we can now state that when the Simplex Algorithm terminates it gives us an optimal solution.

2 Introduction to Duality theorem

Let x_0 be an optimal point. Using the termination condition of Simplex Algorithm we know that cost decreases along the columns of $-A^{'-1}$.In other words,

$$c^T(-A^{'-1}) = (\gamma_1, \gamma_2, ..., \gamma_n), \gamma_i <= 0 \tag{6}$$

or

$$c^T(A^{'-1}) = (y_1, y_2, ..., y_n), y_i >= 0 \tag{7}$$

$$c^T(A^{'-1}) = y^T, y_i >= 0 \tag{8}$$

$$c^T(A^{'-1}A^{'}) = y^T A^{'} \tag{9}$$

$$c^T = y^T A^{'} \tag{10}$$

We observe that at the optimal point the cost vector can be written as a *non-negative* linear combination of the rows of A'. This means that x_0 is optimal iff x_0 is feasible and the cost can be written as a non-negative linear combination of the rows of A'.

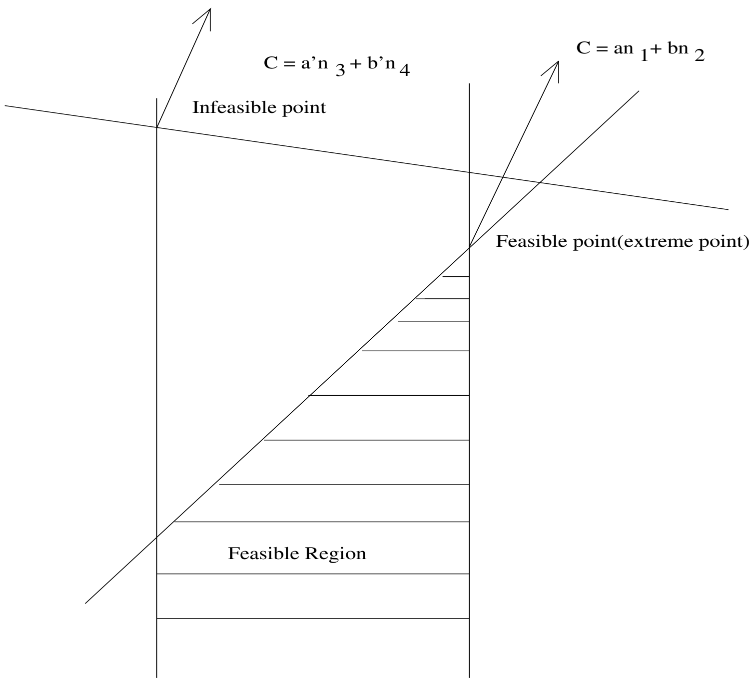

Figure 1: $a, a', b, b' \geq 0$. Cost can be written as a positive linear combination of normals to the hyperplanes.

The rows of A' are also the direction normals to the respective hyperplanes. So a restatement of the above is as follows. Suppose x_0 is an extreme point given by the intersection of n linearly independent hyperplanes then the cost vector can be written as a non-negative linear combination of the normals to these hyperplanes.

Now consider all the points (not necessarily feasible) given by the intersection of n linearly independent hyperplanes where the cost vector can be written as a positive linear combination of the normals. We will show that among such points only the feasible point will have the lowest cost.

Consider the feasible point x_0 and any other point say x satisfying the above requirements, then $x - x_0$ can be written as a positive linear combination of the columns of A' where

$$A'x = b', A^{''}x < b^{''} \tag{11}$$

Note that the cost decreases along the columns of $-A^{'-1}$. Following the steps of the proof of the previous theorem one can show that $x - x_0$ can be written as a non negative linear combination of the columns of $-A^{'-1}$. Since the cost decreases along the columns of $-A^{'-1}$, the cost at x is at least the cost at x_0.

We also note that at such points the cost is $c^T x = y^T A'x = y^T b'$.

This motivates the definition of the following LP called the dual:

$$minimize : y^T b \tag{12}$$

$$A^T y = c \tag{13}$$

$$y >= 0 \tag{14}$$