Let \( x_0 \) be an extreme point. Suppose it is given by:

\[
A' x_0 = b', A'' x_0 < b''
\]

From lecture 11 we know that the neighbours of \( x_0 \) are along the columns of \(-A'^{-1}\).

1 Proof of correctness of Simplex algorithm

**Theorem 1** If the cost decreases along the columns of \(-A'^{-1}\) then \( x_0 \) is optimal.

**Proof:** The columns of \(-A'^{-1}\) span \( R^n \). Let \( x_{opt} \) be an optimal point i.e. \( c^T x_{opt} \geq c^T x_0 \) then we need to show that \( c^T x_{opt} \leq c^T x_0 \) to establish \( c^T x_{opt} = c^T x_0 \) and hence \( x_{opt} = x_0 \). Since the columns of \(-A'^{-1}\) are a basis the vector \( x_{opt} - x_0 \) can be represented as a linear combination of them.

\[
x_{opt} - x_0 = \sum \beta_j (-A'^{-1})^j
\]

Now consider \( A'(x_{opt} - x_0) \)

\[
A' x_{opt} - A' x_0 = \sum \beta_j A'(-A'^{-1})^j
\]

We know that \( A' x_{opt} \leq b' and A' x_0 = b' \) hence \( A'(x_{opt} - x_0) \leq 0 \). Also note that \( A'(-A'^{-1})^j \) is an \( n \times 1 \) vector whose \( j^{th} \) element is -1 and remaining elements are 0. Hence

\[
A' x_{opt} - A' x_0 = \begin{pmatrix}
-\beta_1 \\
-\beta_2 \\
\vdots \\
-\beta_n
\end{pmatrix}
\]

\[
\Rightarrow \forall j \beta_j \geq 0
\]

From the discussion above we infer that \( \beta_j \geq 0 \) for each \( j \).

Now consider

\[
c^T x_{opt} - c^T x_0 = \sum \beta_j c^T(-A'^{-1})^j
\]

Since the cost decreases along the columns of \(-A'^{-1}\) we have \( c^T(-A'^{-1})^j \leq 0 \) and since \( \beta_j \geq 0 \) we conclude that \( \sum \beta_j c^T(-A'^{-1})^j \leq 0 \)

Hence \( c^T x_{opt} \leq c^T x_0 \) but we know that \( c^T x_{opt} \geq c^T x_0 \) and \( c^T x_{opt} = c^T x_0 \).

\( \square \)
Note: Using the above theorem we can now state that when the Simplex Algorithm terminates it gives us an optimal solution.

2 Introduction to Duality theorem

Let \( x_0 \) be an optimal point. Using the termination condition of Simplex Algorithm we know that cost decreases along the columns of \(-A'^{-1}\). In other words,

\[
c^T(-A'^{-1}) = (\gamma_1, \gamma_2, ..., \gamma_n), \gamma_i \leq 0
\]

or

\[
c^T(A'^{-1}) = (y_1, y_2, ..., y_n), y_i \geq 0
\]

\[
c^T(A'^{-1}A') = y^T A'
\]

\[
c^T = y^T A'
\]

We observe that at the optimal point the cost vector can be written as a non-negative linear combination of the rows of \( A' \). This means that \( x_0 \) is optimal iff \( x_0 \) is feasible and the cost can be written as a non-negative linear combination of the rows of \( A' \).

Figure 1: \( a, a', b, b' \geq 0 \). Cost can be written as a positive linear combination of normals to the hyperplanes.

The rows of \( A' \) are also the direction normals to the respective hyperplanes. So a restatement of the above is as follows. Suppose \( x_0 \) is an extreme point given by the intersection of \( n \) linearly independent hyperplanes then the cost vector can be written as a non-negative linear combination of the normals to these hyperplanes.
Now consider all the points (not necessarily feasible) given by the intersection of \( n \) linearly independent hyperplanes where the cost vector can be written as a positive linear combination of the normals. We will show that among such points only the feasible point will have the lowest cost.

Consider the feasible point \( x_0 \) and any other point say \( x \) satisfying the above requirements, then \( x - x_0 \) can be written as a positive linear combination of the columns of \( A' \) where

\[
A' x = b', A'' x < b''
\]  
(11)

Note that the cost decreases along the columns of \( -A'^{-1} \). Following the steps of the proof of the previous theorem one can show that \( x - x_0 \) can be written as a non negative linear combination of the columns of \( -A'^{-1} \). Since the cost decreases along the columns of \( -A'^{-1} \), the cost at \( x \) is at least the cost at \( x_0 \).

We also note that at such points the cost is \( c^T x = y^T A' x = y^T b' \).

This motivates the definition of the following LP called the dual:

\[
\begin{align*}
\text{minimize : } & y^T b \\
A^T y & = c \\
y & \geq 0
\end{align*}
\]  
(12-14)