CS 435 : Linear Optimization — Fall 2006

Lecture 13: Proof of correctness of the Simplex algorithm and introduction to duality

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From the previous lectures, we know the following fact that, if x_0 is an extreme point given by

$$A^{'} x_0 = b^{'} \tag{1}$$

$$A^{''} x_0 < b^{''} \tag{2}$$

then the neighbours of x_0 are along the columns of $-A^{'^{-1}}$.

Theorem 1 *If the cost decreases along the columns of $-A^{'^{-1}}$, then x_0 is optimal.*

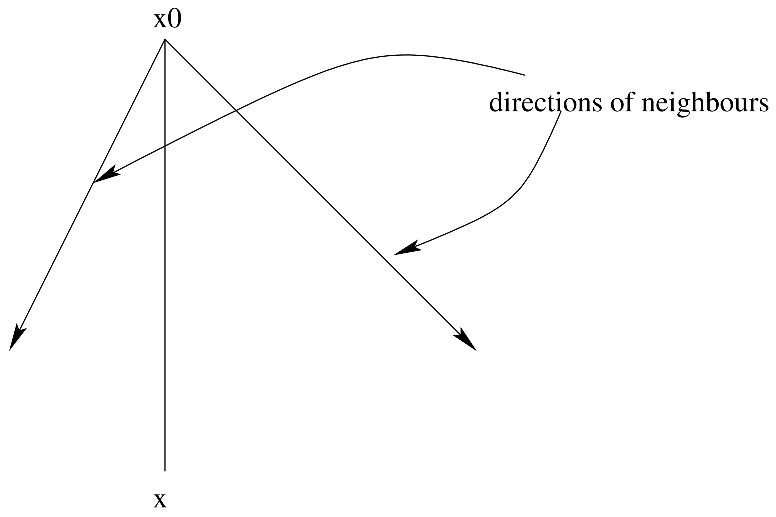

Figure 1: Representation of (x-x0)

Proof: As $A^{'}$ has full rank, $-A^{'^{-1}}$ also has full rank. Thus the columns of $-A^{'^{-1}}$ form a basis of $\Re^n$. Hence any vector can be written as a linear combinations of these n columns.

$$x - x_0 = \sum (\beta_j)(-A^{'^{-1}})^j \tag{3}$$

Premultiplication with $A^{'}$

$$A^{'} x - A^{'} x_0 = \sum \beta_j A^{'} (-A^{'^{-1}})^j \tag{4}$$

Since x is feasible $A^{'} x <= b^{'}$. Also $A^{'} x_0 = b^{'}$. Thus the L.H.S. of equation 4 is a vector each of whose components is at most zero. The R.H.S. is the vector $(\beta_1, \beta_2, ..., \beta_n)^T$. This implies that $\beta_j >= 0$ for all j. Hence $x - x_0 = \sum \beta_j (-A^{'^{-1}})^j$ where $\beta_j >= 0$. Premultiplication with c^T gives

$$c^T x - c^T x_0 = \sum \beta_j c^T (-A^{'^{-1}})^j \tag{5}$$

Since $\beta_j >= 0$ and $c^T (-A^{'^{-1}})^j <= 0$ for all j, therefore $c^T x <= c^T x_0$. Hence x_0 is optimal. □

Discussion:
Let x_0 be the optimal point and

$$A^{'} x_0 = b^{'} \tag{6}$$

and

$$A^{''} x_0 < b^{''} \tag{7}$$

The cost decreases along the columns of $-A^{'-1}$. This can be written as

$$c^T A^{'-1} = (y_1, y_2, ..., y_n) \text{ where } y_i >= 0, \forall i \tag{8}$$

This means that the cost vector is a positive linear combination of the normals to the hyperplanes.
Consider all points (not necessarily feasible) given by n linearly independent hyperplanes, where the cost vector can be written as a positive linear combination of the normals. These points are the candidate maximums.

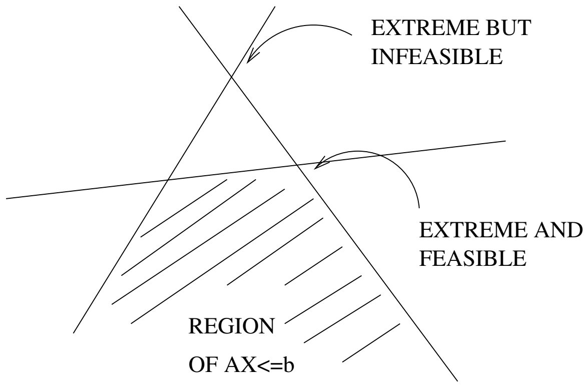

Figure 2: 2-D example of extreme but infeasible points

Take any such point x say
Then $c^T x >= c^T x_0$, where x_0 is the optimal point. Clearly this can be written as another linear program in the following manner - As $c^T x = y^T A^{'} x = y^T b^{'}$, thus dual linear program is

$$\begin{gathered} \min y^T b^{'} \\ A^T y = c \\ y >= 0 \end{gathered}$$