

Lecture 15: Complementary slackness, infeasible primal \Leftrightarrow unbounded dual, infeasible dual \Leftrightarrow unbounded primal

Lecturer: *Sundar Vishwanathan*
COMPUTER SCIENCE & ENGINEERING

Scribe: *Aman Parnami*
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

1 Revision

Following were done in previous lecture

- Primal and Dual

Primal	Dual
$\max c^T x$	$\min y^T b$
$Ax \leq b$	$A^T y = c$
	$y \geq 0$

- Proof of existence of a dual for a feasible and bounded primal.
- THEOREM 1 (DUALITY THEOREM) *If both the primal and the dual of an LP are feasible, then their optimum value coincide i.e*

$$c^T x_0 = y_0^T b, \text{ where } x_0, y_0 \text{ are optimals for Primal and Dual resp.}$$

2 Optimality Condition

Consider an x_0 such that $\{A\}_{m \times n} \{x_0\}_{n \times 1} \leq \{b\}_{m \times 1}$ and a $\{y_0\}_{m \times 1}$ such that $(y_0)_i \geq 0$ and $A^T y_0 = c$

Then the following statements are equivalent.

1. x_0 and y_0 are optimal $\Leftrightarrow c^T x_0 = y_0^T b$
2. x_0 and y_0 are optimal $\Leftrightarrow (y_0)_i > 0 \Rightarrow A_i x_0 = b_i$

Statement 2 is called **Complementary Slackness**. Geometrically, it means hyperplanes corresponding to (+)ve $(y_0)_i$'s intersect at x_0 . In the other case, $(y_0)_i = 0 \Rightarrow A_i x_0 < b_i$ i.e A_i corresponds to a hyperplane which does not pass through x_0 .

PROOF: We use 1 as definition of Optimality.

• **2** \Rightarrow **1**

$$\begin{aligned}
 y_0^T b &= \sum_{j=1}^m y_{0j} b_j \\
 &= \sum_{j=1}^m y_{0j} (A_j x_0) \quad (\text{using } A_i x_0 = b_i) \\
 &= \sum_{j=1}^m y_{0j} \left(\sum_{i=1}^n A_{ji} x_{0i} \right) \\
 &= \sum_{i=1}^n x_{0i} \left(\sum_{j=1}^m A_{ji} y_{0j} \right) \quad (\text{using } A^T y_0 = c). \\
 &= x_0^T c \\
 &= c^T x_0
 \end{aligned}$$

• **1** \Rightarrow **2**

$$\begin{aligned}
 y_0^T b &= x_0^T c \\
 &= \sum_{i=1}^n x_{0i} \left(\sum_{j=1}^m A_{ji} y_{0j} \right) \quad (\text{using } A^T y_0 = c) \\
 &= \sum_{j=1}^m y_{0j} \left(\sum_{i=1}^n A_{ji} x_{0i} \right) \\
 &= \sum_{j=1}^m y_{0j} (A_j x_0) \\
 &\leq \sum_{j=1}^m y_{0j} b_j \quad (\text{using } y_{0j} \geq 0 \text{ and } A_j x_0 \leq b_j) \\
 &= y_0^T b
 \end{aligned}$$

\Rightarrow for non-zero (+)ve $(y_0)_i$'s, $A_i x_0 = b_i$ otherwise inequality remains leading to contradiction.

□

3 Next Class

Following questions were raised at the end of class and will be fully discussed in next lecture :

- What if any of the Primal or Dual is **infeasible** ?
Infeasibility means no point satisfies all the constraints.
- What if cost is **unbounded** ?
Unbounded here means no optimum value exists or that there are feasible points with arbitrary large cost.
Ex. $\max -x, x \leq 10$
Here optimum value for cost = $-x$ can't be found with given constraints.