Lecture 15: Complementary slackness, Duality Theeorem

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1 The Duality Theorem

Primal	Dual
$\max c^T x$	$\min y^T b$
$Ax \leq b$	$A^T y = c$
	$y \geq 0$

Theorem 1 *If both the primal and the dual of an LP are feasible, then their optimum values coincide*

If primal LP is feasible and bounded, there is some optimal point x_o. As discussed in the last class we can construct a $y \geq 0$ such that $A^T y = c$ and $y^T b = c^T x$. This means that the dual is feasible and the optimum values coincide.

2 Optimality Condition

Consider an x_0 such that $Ax_0 \leq b$ and a y_0 such that $y_0 \geq 0$ and $A^T y_0 = c$. Then the statements

1. x_0 and y_0 are optimal respectively for the primal and dual if and only if $c^T x_0 = y_0^T b$.

2. x_0 and y_0 are optimal respectively for the primal and dual if and only if $(y_0)_i > 0 \Rightarrow A_i x_0 = b_i$.

are equivalent.
Proof: We prove 1 using 2

$$
\begin{aligned}
y_0^T b &= \sum_{j=1}^{m} y_{0j} b_j \\
&= \sum_{j=1}^{m} y_{0j} (A_j x_0) \qquad \text{(using 2)} \\
&= \sum_{j=1}^{m} y_{0j} \left(\sum_{i=1}^{n} A_{ji} x_{0i} \right) \\
&= \sum_{i=1}^{n} x_{0i} \left(\sum_{j=1}^{m} A_{ji} y_{0j} \right) \\
&= x_0^T c \qquad \text{(using } A^T y_0 = c \text{)} \\
&= c^T x_0
\end{aligned}
$$

Proving 2 using 1

$$
\begin{aligned}
c^T x_0 &= x_0^T c \\
&= \sum_{i=1}^{n} x_{0i} \left(\sum_{j=1}^{m} A_{ji} y_{0j} \right) \\
&= \sum_{j=1}^{m} y_{0j} \left(\sum_{i=1}^{n} A_{ji} x_{0i} \right) \\
&= \sum_{j=1}^{m} y_{0j} (A_j x_0) \\
&\leq \sum_{j=1}^{m} y_{0j} b_j \qquad \text{(using } y_{0j} \geq 0 \text{ and } A_j x_0 \leq b \text{)} \\
&= y_0^T b
\end{aligned}
$$

But we know that $c^T x_0 = y_0^T b$, hence, $y_0 > 0 \Rightarrow A_i x_0 = b_i$ This condition is called complementary slackness. □

3 Infeasibility and Unboundedness

Every LP problem is either feasible or infeasible. Now, the feasible problems either have a solution or are unbounded.
Infeasibility implies that there is no solution to Ax ≤ b. Example of an unbounded LP is

$$
\begin{aligned}
&\max\ (-x) \\
&\quad x \leq 10.
\end{aligned}
$$