Lecture 16: Another way of looking at duality

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In this lecture we will see another way of interpreting duality. However, before that, let us clarify some doubts on duality that few students seemed to have in the last class.

Let us consider the following primal-dual pair.

$$\begin{aligned} \text{P}: \quad & \max\ c^T x \\ & \text{s.t. } Ax \le b \end{aligned} \qquad \begin{aligned} \text{D}: \quad & \min\ y^T b \\ & \text{s.t. } A^T y = c \\ & y \ge 0 \end{aligned}$$

All that the duality theorem states is the following.

- "P is feasible and has a finite maximum" $\Rightarrow$ "D is feasible and the two optimum values coincide".
- "P is infeasible and D is feasible" $\Rightarrow$ "D is unbounded".
- "P feasible and unbounded" $\Rightarrow$ "D is infeasible".

Solve the following exercises.

EXERCISE 1 Construct a P-D pair where both are infeasible.

EXERCISE 2 Show that dual of D is actually P.

EXERCISE 3 Comment about the feasible regions of P & D.

1 Another way of looking at duality

Let us consider the following LP problem.

$$\begin{aligned} \max\ & 14x_1 + 7x_2 + 22x_3 + 10x_4 \\ \text{s.t. }\ & 10x_1 + 3x_2 + 10x_3 + 7x_4 \le 20 && (1) \\ & 3x_1 - 12x_2 - 13x_3 + 14x_4 \le 35 && (2) \\ & 4x_1 + 4x_2 + 12x_3 + 3x_4 \le 4 && (3) \end{aligned}$$

If we look at the above equations closely we can see that an upper bound of the objective is 24. This is because if we simply add (1) and (3), we get $14x_1 + 7x_2 + 22x_3 + 10x_4 \le 24$. We could have multiplied the above equations by any non-negative factors (if we multiply by negative factors, the direction of the inequalities changes which we do not want) and then added to get an upper bound of the objective. This observation gives us another way of solving the LP.

Let us have the following general LP. Say it the primal (P).

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t. }\ & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \quad \} \times y_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \quad \} \times y_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \quad \} \times y_m \end{aligned}$$

Suppose we are able to find out non-negative multipliers $y_1, y_2, \cdots, y_m$ such that, when the equations are multiplied by the respective multipliers and added together, we get the objective functions on the left hand side. That is,

$$
\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &= c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &= c_2 \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &= c_n \\
y_i &\geq 0 \qquad\qquad \forall i = 1..m
\end{aligned}
$$

Then the sum of the right hand sides i.e. $b_1y_1 + b_2y_2 + \cdots + b_my_m$ gives an upper bound of the primal. Now consider all possible sets of multipliers satisfying the above requirement. Consider the upper bound given by each such set. The least among the upper bounds gives the optimum of the primal. Hence, the optimum objective of the primal is the same as the optimum objective of the following LP. Say it the dual (D).

$$
\begin{aligned}
\min\ & b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{s.t.}\ & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m = c_1 \\
& a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m = c_2 \\
& \vdots \\
& a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m = c_n \\
& y_i \geq 0 \qquad\qquad \forall i = 1..m
\end{aligned}
$$

Let us explore all cases of the duality theorem from this new perspective.

1. "P is feasible and has a finite maximum". It easily follows from the above discussion that "D is feasible and the two optimum values coincide".

2. "P feasible and unbounded". In this case, we wont be able to find a set of multipliers with the requirement stated in the above discussion. The reason is as follows. Suppose we are able to find some set of multipliers $y_1^*, y_2^*, \cdots, y_m^*$. Then the objective of P should be bounded from above by $b_1y_1^* + b_2y_2^* + \cdots + b_my_m^* = U$, some finite quantity. But, that contradicts the fact that P is unbounded. Hence, "D is infeasible".

3. "P is infeasible and D is feasible". Then D is unbounded. The reason is as follows. Suppose, D is bounded. Then D is both feasible and bounded. Again, we have seen that the dual of D is P. Now, taking D as the primal, by the statement already proved in part 1, P is feasible. But that contradicts the fact that P is infeasible. Hence "D is unbounded".