

Lecture 17: Matching in Bipartite Graphs

Lecturer: *Sundar Vishwanathan*
COMPUTER SCIENCE & ENGINEERING

Scribe: *Ramesh Nidadavolu*
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

A set of disjoint edges in G is called a *matching*. *Maximum matching* in a graph G is a *Matching* of maximum size.

First we discuss a few details. If M is *matching* and M_0 is a *maximum matching* then $M \Delta M_0$ ¹ has connected components which are either even cycles or simple paths. Also in a odd path the first edge in the path cannot be an edge from the matching M , else we could interchange the M_0 edges and the M edges in this path and increase the size of the matching M_0 by 1.

Another interesting thing to note is that if $A \Delta B = C$ then $A \Delta C = B$. (Proof is simple. Draw a venn diagram to see that it should hold)

We also state that $M \Delta M_0$ has atleast one component which is a path of odd length.(Proof is easy. Solve the exercises below to figure out the proof)

EXERCISE 1 If M is a matching of size $|M_0| - 1$ then there exists an odd length path in $M \Delta M_0$.

EXERCISE 2 For any matching M there is atleast one odd length path in $M \Delta M_0$. (Hint: If all components would be even then $|M| = |M_0|$?)

Also as we pointed out earlier any odd length path has to start with an edge from M_0

Now given a graph G and a matching M we define a *augmenting path* as a path in G whose start and end edges are not in M and alternate edges are in M along the path.(so an *augmenting path* would have odd number of edges with more unmatched edges than matched edges.)

Now we prove that if a matching M is not a maximum matching there must exist an *augmenting path* wrt to M . (Actually a stronger result is proven below)

1 Main result

We prove the following result: *A matching M is maximum iff there does not exist any augmenting path in G .*

If the matching is maximum and there exists an *augmenting path* P then $M \Delta P$ would be a matching of greater size.Contradiction. Also If the matching is not maximum then there must exist an odd length path given by $M \Delta M_0$ (as proven earlier).

Thus proven.

2 Why this result?

The idea is that in searching for a matching that is maximum we can start of with some arbitrary matching M and then look for *augmenting paths* and then construct a new matching $M \Delta P$ which would be of a size greater than the matching we started of with. Thus repeatedly looking for *augmenting paths* and incrementing our matching size we hope to end up with a maximum matching. More on this in the next lecture.

¹ $A \Delta B = (A - B) \cup (B - A)$