Lecture 22: Using LP techniques to design algorithms for combinatorial problems

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1 Integer Linear Programs

An *Integer Linear Program* is a linear program where the variables are constrained to take integer values only. An ILP is formulated as

$$\max\ c^T x$$
$$Ax \leq b$$
$$x_i \text{ is integral } \forall i.$$

2 Recap of primal,dual and complementary slackness

Recapitulate that for an LP

$$\max\ c^T x$$
$$Ax \leq b$$

the dual is

$$\min\ y^T b$$
$$A^T y = c$$

Complementary slackness implies that the optimums of the primal and dual coincide i.e. $c^T x_o = (y_o)^T b$ and where $y_{oi} > 0$ in the dual $A_i x_0 = b_i$ in the primal and vice versa.

3 Formulation Of Minimum Spanning Tree as an ILP

A *spanning tree* is a subgraph of a connected,undirectional graph that connects all vertices and is a tree. The minimum spanning tree is a spanning tree with minimum cost or sum of weight of edges part of the tree. The corresponding ILP may be formulated taking as variables x_e ,one for each edge where a value of 1 for x_e indicates that the edge is part of the tree and value of zero indiacates that the edge is not part of the spanning tree. The cost then is $\sum x_e c_e$ where c_e is the weight of the edge. The ILP formulation then is

$$\min\ c^T x = \sum x_e c_e$$

Constraints :

$$\begin{gathered} \forall \text{ partitions } \pi \\ \sum_{e crosses \pi} x_e \geq \#(\pi) - 1 \\ 0 \leq x_e \leq 1 \\ x_e \text{ is integral} \end{gathered}$$

where $\#(\pi)$ is the number of parts in the partition π. The first constraint basically represents that the graph is connected.

Now we drop the integral constraint and write the dual of the resulting LP. The dual is

$$\max(\sum_{\pi} y_\pi(\#(\pi) - 1))$$

$$\begin{gathered} \forall e : \sum_{e crosses \pi} y_\pi \leq c_e \\ y_\pi \geq 0 \end{gathered}$$