Lecture 23: Primal-dual algorithm for MST (contd..)

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1 A brief recap of last lecture

The *relaxed* LP formulation (i.e. after removing the constraints x_e *integer*$, x_e \leq 1$) for the MST of a graph $G(V, E)$ is the following:

$$\begin{aligned} \min \quad & \sum_e c_e x_e \\ \text{s.t.} \quad & \sum_{e \text{ crosses } \pi} x_e \geq \#(\pi) - 1 \quad \forall \pi \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

where

- $c_e \geq 0$ is the weight of the edge $e \in E$
- π denotes a partition of the vertex set V and can be represented as the set of disjoint subsets of the vertex set V, i.e. $\pi \equiv \{p_1, p_2, \ldots, p_k\}, p_i \subset V, \cup p_i = V, p_i \cap p_j = \phi$. # is the number of 'parts' in π i.e. $= k$
- e crosses π means one end vertex of e belongs to p_i and other end belongs to $p_j, i \neq j$.

The dual LP is given by

$$\begin{aligned} \max \quad & \sum_\pi y_\pi (\#(\pi) - 1) \\ \text{s.t.} \quad & \sum_{e \text{ crosses } \pi} y_\pi \leq c_e \quad \forall e \in E \\ & y_\pi \geq 0 \quad \forall \pi \end{aligned}$$

2 The algorithm for MST

The algorithm that we designed is the following

Algorithm 1 MST algorithm developed using LP duality

```
Initialize all y_π ← 0
Set i ← 0, π_0 ← the partition in which each vertex is a part by itself
repeat
    Raise y_{π_i} until dual constraint corresponding to some edge e becomes tight
    Set x_e ← 1
    Set π_{i+1} ← the partition got from π_i by merging end points of e
    Set i ← i + 1
until edges with x_e = 1 forms a spanning tree
```

We start with any feasible solution for the dual LP and maintain feasibility in both primal and dual LP's. In each iteration, we try to improve the dual solution. For this, we raise the value of the dual variables one by one. If a constraint becomes tight for an edge we donot raise the dual variable for partitions which this edge crosses. Note that implicitly we are doing what problem 2 in Quiz 2 dictates.

After each iteration, we take the edges for which the constraints are equalities. If this set of edges yield a connected graph, we are done. A spanning tree will yield a primal feasible solution. Note the use of complementary slackness.
We essentially start with zero partition (i.e. an empty tree initially) and add edges as we go along until dual constraint corresponding to some edge e becomes tight. We then include e into our solution by setting $x_e = 1$ and partition is updated by merging end points of e. The algorithm terminates with a spanning tree having edges with $x_e = 1$ of the input graph.

3 Proof of optimality

We can prove that the above algorithm gives optimum solution by proving that cost of primal and dual are same and they both are feasible.

Cost of primal is given by:

$$
\begin{aligned}
& \sum_e c_e \quad \text{where e is chosen by the algorithm} \\
= & \sum_e \sum_{\substack{\pi \\ \text{e crosses } \pi}} y_\pi \\
= & \sum_\pi \sum_{\substack{e \\ \text{e crosses } \pi}} y_\pi \\
= & \sum_\pi y_\pi \sum_{e \text{ crosses } \pi} 1 \\
= & \sum_\pi y_\pi \ (\text{\# of edges chosen which crosses } \pi)
\end{aligned}
$$

$$\Downarrow$$

$$(\#(\pi) - 1) \leftarrow \text{where } y_\pi > \ 0$$

This is exactly the cost of the dual. So we have proved that the cost of primal is same as the cost of the dual $\Rightarrow$ both of them are optimal

4 Some comments and observations

- Optimality can be proved by using complimentary slackness
- This LP has an integral solution which is optimal
- Number of constraints is exponential(becaouse number of partitions is exponential)
- The edges which become tight are in increasing order of weights

5 Exercises

- Run the algoritm with π restricted to 2. Manipulate the dual variables so that you end up with Kruskal's algorithm. Apply the above proof of correctness, you will get struck.

- $\forall$ n, find an example where optimum of the LP is less than the weight of the minimum spanning tree.
- Prove that if we add a new constraint

$$\sum x_e = n - 1,$$

  there exist an integral solution to the LP
- Observe that the way we choose y_π corresponds to the last problem of quiz 2.