# Lecture 24: ILP formulations for SAT and shortest path problem

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## 1 Formulating ILP for SAT

An instance of a SAT problem is a Boolean expression written using only AND, OR, NOT, variables, and parentheses in its CNF form. The question is: given the expression, is there some assignment of TRUE and FALSE values to the variables that will make the entire expression true?

**Input:** A set of boolean variables $z_1$,$z_2$,$z_3$, ... ,$z_{n-1}$,$z_n$, and a boolean expression in CNF for which a satisfying assignment is to be found.

$$\begin{array}{c}(x_{11} \vee x_{12} \vee x_{13} \vee \ldots x_{1i_1})\wedge \\ (x_{21} \vee x_{22} \vee x_{23} \vee \ldots x_{2i_2})\wedge \\ (x_{31} \vee x_{32} \vee x_{33} \vee \ldots x_{3i_3})\wedge \\ \ldots \\ (x_{k1} \vee x_{k2} \vee x_{k3} \vee \ldots x_{ki_k})\end{array}$$

Where each $x_{ij}$ is either a variable($z_t$), or its negation ($\bar{z}_t$)

**Output:** An assignment for each $z_t$, such that the above boolean expression evaluates to true.

**Solution:** ILP formulation of any problem has three parts

- **Variables**
  Here we attach each an integer variable $y_t$ to each boolean variable $z_t$, which represents the assignment of true or false.
  $$y_t = \begin{cases} 0 & \text{if false,} \\ 1 & \text{if true} \end{cases}$$
  $y_t = 1$ - $y_t$ if $z_t$ is in negated $0 \leq y_t \leq 1$
- **Constraints**
  Constraints will be given by the conditions necessary for each clause of the expression to be true.
  $$\begin{array}{c} y_{11} + y_{12} + y_{13} + ... y_{1i_1} \geq 1 \\ y_{21} + y_{22} + y_{23} + ... y_{2i_2} \geq 1 \\ y_{31} + y_{32} + y_{33} + ... y_{3i_3} \geq 1 \\ ... \\ y_{k1} + y_{k2} + y_{k3} + ... y_{ki_k} \geq 1 \end{array}$$
  Where $y_{ij}$ represents the variable $y_t$ attached to the corresponding $z_t$.
- **Cost**
  Here cost is immaterial as any feasible solution gives a satisfying assignment.

## 2 Formulating ILP for shortest path problem

**Input:** A directed graph with positive integer weights ($w_{uv}$) and two vertices $s$ and $t$ from its vertex set.

**Output:** Shortest/min weight path from $s$ to $t$

- **Variables**

  Attach one integer variable to each edge $x_{uv}$ which indicates whether the edge $(u, v)$is chosen or not.

$$x_{uv} = \begin{cases} 1 & \text{if edge } (u, v) \text{ chosen,} \\ 0 & \text{o/w} \end{cases}$$

  $0 \leq x_{uv} \leq 1$

- **constraints**

  If there is a path from s to t in the graph then for any two partitions, such that one of them contains s and other contains t, there must be an edge leaving the partition containg s to the partition containing t ( ∃ u ∈ U and v ∈ V-U s.t u and v are connected).

$$\forall U \subset V, s \in U, t \notin U, \forall u \in U \text{ and } v \notin U \Sigma x_{uv} \geq 1$$

- **Cost** We want to minimize the weight of the path from s to t, ie the sum of all edges in the path should be minimum,

$$\min(\Sigma x_{uv} w_{uv})$$

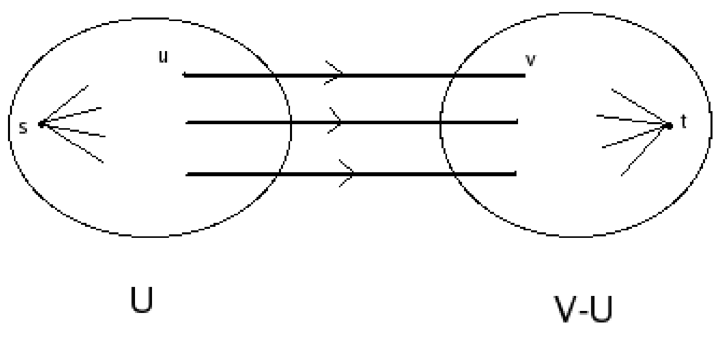


Figure 1: Example of a partition

This has an exponential number of constraints. In next class we will see how to write this using a polynomial number of constraints.