

Lecture 25: ILP formulations for SAT and shortest path problem (contd.)Lecturer: *Sundar Vishwanathan*
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In this lecture we will formulate Integer Linear Programs(ILPs) for some of the problems discussed in the tutorial.

1 SAT

A SAT problem can be defined as follows:

- **Input:** An expression of the form $(x_i \vee \bar{x}_j \vee \dots) \wedge (\dots \vee \dots \vee \dots) \wedge \dots$, where a variable x_k can take value T or F.
- **Output:** Assignment of TRUE/FALSE values to literals x_i s) for the expression to be true or NOT SOLVABLE if such an assignment is not possible.

Our task is to formulate this problem as an ILP:

- **Variables:** There will be one variable y_i per boolean variable x_i

$$y_i = \begin{cases} 0 & \text{if } x_i \text{ is FALSE} \\ 1 & \text{if } x_i \text{ is TRUE} \end{cases} \quad (1)$$

$$0 \leq y_i \leq 1, y_i \text{ is integral}$$

This ensures that y_i is either 0 or 1.

Note: If we choose y_i s to be -1 and 1 in place of 0 and 1, formulation in the above form is difficult.

- **Inequalities:** Consider a clause $x_i \vee x_j$. This will be true if either $x_i = 1$ or $x_j = 1$ or both. Clearly, the inequality will be $y_i + y_j \geq 1$. Similarly, for the clause $x_i \vee \bar{x}_j$, the inequality will be $y_i + 1 - y_j \geq 1$ or $y_i - y_j \geq 0$.
- **Inequality Matrix:** The inequality obtained from each clause will correspond to one row of the matrix.
- **Cost function:** Immaterial in this case as long as we can find feasible y_i s satisfying $Ay \leq b$, $0 \leq y_i \leq 1$, y_i is integral.

2 Shortest Path

A Shortest Path problem can be defined as follows:

- **Input:** A Directed Graph with edge weights as positive integers and two special vertices s and t .
- **Output:** Set of edges which form a shortest path between s, t .

Our task is to formulate this problem as an ILP:

2.1 Formulation 1

- **Variables:** Clearly the output is a subset of the original edge set. So there will be one variable per edge(e) between two vertices u and v , x_{uv}

$$x_{uv} = \begin{cases} 0 & \text{if the edge from vertex } u \text{ to vertex } v \text{ is not chosen for the shortest path} \\ 1 & \text{if the edge from vertex } u \text{ to vertex } v \text{ is chosen for the shortest path} \end{cases} \quad (2)$$

$$\begin{aligned} 0 &\leq x_{uv} \leq 1, \\ x_{uv} &\text{ is integral} \end{aligned}$$

Define

c_{uv} - cost of the edge from vertex u to vertex v

- **Cost function:** Clearly the cost function in this case is

$$\min \sum x_{uv} c_{uv}$$

Before we formulate the inequality let us state a lemma.

LEMMA 1 *Let V be a set of vertices and let s, t be the two vertices for which the shortest path has to be found. For every $U \subset V$ s.t. $s \in U, t \notin U$ at least one of the arcs coming out of U must be selected.*

- **Constraints:** From the above lemma we can see that the following inequalities force a path from s to t

$$\begin{aligned} \forall \text{ subsets } U \text{ of } V, \text{ s.t. } s \in U, t \notin U \\ \sum_{u \in U, v \notin U} x_{uv} \geq 1 \end{aligned}$$

2.2 Formulation 2

In this formulation we use a polynomial number of constraints

- **Constraints:** Clearly there will be only one edge coming out of the starting vertex s in the shortest path and there will be only one vertex coming into the terminating vertex e . This is represented as follows

$$\begin{aligned} \sum_v x_{sv} - \sum_p x_{ps} &= 1 \\ \sum_u x_{ut} - \sum_q x_{tq} &= 1 \end{aligned}$$

For all the other vertices (including the ones which are not in the shortest path) the number of edges (of the shortest path) coming in should be equal to the number of edges going out.

$$\forall u \neq s, e \quad \sum_p x_{pu} - \sum_q x_{uq} = 0$$

$$x_{uv} = \begin{cases} 0 & \text{if the edge from } u \text{ to } v \text{ is not chosen for the shortest path} \\ 1 & \text{if the edge from } u \text{ to } v \text{ is chosen for the shortest path} \end{cases} \quad (3)$$

$$0 \leq x_{uv} \leq 1, x_{uv} \text{ is integral}$$

Clearly each of the vertices in the Vertex Set will contribute to one row of the matrix.