Lecture 26: Primal-Dual Algorithm for Shortest Path Problem

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1 Overview

In the previous lecture, we saw the formulation of the Integer Linear Program for the shortest path algorithm. In this lecture we formulate and solve the dual.

2 The formulation of the shortest path problem

Input:

A directed graph with positive integer weights, $s, t \in V$

Output:

Shortest path from s to t

Variables:

We choose one variable per edge, x_e. If x_e is picked, $x_e = 1$ else $x_e = 0$. w_e is the value of the weight of edge e

Cost:

The cost function is min $\sum_{u,v} x_{uv} w_{uv}$

Constraints:

1. $\sum_u x_{su} - \sum_v x_{vs} = 1$ (The difference in the number of edges leaving s and entering into s is 1)
2. $\sum_v x_{vt} - \sum_u x_{tu} = 1$ (The difference in the number of edges entering into t and leaving t is 1)
3. $\forall p \in V - \{s, t\}$ $\sum_q x_{pq} - \sum_r x_{rp} = 0$ (For all other vertices, edges leaving them is equal to the edges entering into them)
4. $0 \leq x \leq 1,$ x is integer. We drop integer and upper bound constraints on x as in the case of MST.

3 The Dual

The resulting dual of the primal will have one variable for each vertex in the graph.

$$\max y_s - y_t$$
$$\forall u, v \ y_u - y_v \leq w_{uv}$$

Algorithm

We try to devise a strategy to solve for the values of y subject to the constraint that $y_u - y_v \leq w_{uv}$ and maximising the objective function $y_s - y_t$. The general scheme while trying to go about finding a solution would be as follows:

1. Start with a feasible dual solution.
2. Always maintain dual feasibility.
3. Try and improve.

Now, if y is a solution, $y \pm \delta$ will also be a solution since the constraints would be satisfied. So, the value of y_t is arbitrarily set to 0. Further, we start by setting $y_u = 0 \, \forall u$ and we try to improve the solution thereupon. We will facilitate this process using the marbles and strings analogy described below.

Marbles and Strings analogy

Consider the vertices of the graph as marbles and the edges as inextensible strings (wound to the marbles) with lengths equal to the value of the weights of the corresponding edges. The value y_u for vertex u is interpreted as the distance of node u from the node t ($y_t = 0$, as assumed before). We start by assigning the values of $y_u = 0 \, \forall u$, the situation which is represented by all the marbles being at the position of marble t. Thus, all the strings are slack at this initial point.

Now, maintaining the value of y_t at 0 (keeping the position of marble t fixed), the values of all other $y's$ are raised *simultaneously* (this ensures that the dual feasibilty is maintained). This would be analogous to pulling all the marbles simultaneously (in a straight line) away from marble t. This is done till one of the strings joining marble t becomes taut. Suppose it is the string that joins marble m to t that has become taut. At this point, $y_m = w_{mt}$, and we can no longer pull this particular marble away from t if dual feasibilty is to be maintained. So we fix the position of this marble m now and start pulling all the other marbles till another string becomes taut. We again fix the position of that corresponding marble and keep pulling the other marbles. This process is continued until we find that in due time one of the strings attached to marble s has become taut at which point that has to be fixed. At this point, all the strings corresponding to the edges that belong to the shortest path are taut. Among the edges that have become tight, there are some edges which are not part of the shortest path. These can be removed using the *reverse delete procedure* described below.

Reverse Delete Procedure:

Consider edges in reverse order of them becoming tight. Delete the edge and now check if a path exists from s to t. If a path does exist, then this edge is not a part of the shortest path and it should be removed from the solution set.

Proof that the value of primal optimal is equal to the value of dual optimal:

$$\sum_{(u,v)\, chosen} w_{uv} = \sum_{(u,v)\, chosen} (y_u - y_v) = y_s - y_t$$