Lecture 27: Primal-dual algorithm for matching in bipartite graphs

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1 Primal Dual Method

Primal	Dual
$\max c^T x$	$\min y^T b$
$Ax \leq b$	$A^T y = c$
	$y \geq 0$

Now consider the following unweighted LP for the above primal.

$$\max c^T y$$
$$A' y \leq 0$$

$$\text{Let } X' = X + \epsilon y$$
where X satisfies the primal and ϵy satisfies the above unweighted LP

Then X' satisfies primal as well and the value of its objective function is greater than that for X. This gives us a method for solving the primal. We keep on increasing X_i's in the primal by y till the value of $c^T y$ becomes 0. In the next section, we frame bipartite matching as an ILP.

2 ILP for Bipartite Matching

- **Primal formulation** Let the variable X_{uv} be defined as follows

$$X_{uv} = \begin{cases} 0 & \text{if edge (u,v) is not matched,} \\ 1 & \text{if edge (u,v) is matched} \end{cases}$$

  Then the ILP can be formulated as

$$\max \sum X_{uv}$$
$$\forall u \sum X_{uw} \leq 1$$
$$\forall v \sum X_{pv} \leq 1$$
$$\forall u, v \; 0 \leq X_{uw} \leq 1, X_{uv} \text{ is integral}$$

  Now, as before, we remove the conditions $X_{uw} \leq 1$ and X_{uv} is integral to get LP correspoding to the above ILP[1].

- **Dual formulation** The dual will have variables Y_u and Z_v corresponding to each vertex (Figure 1).

[1]Even after removing the integrality constraints, the solution after relaxation is same as that without relaxation

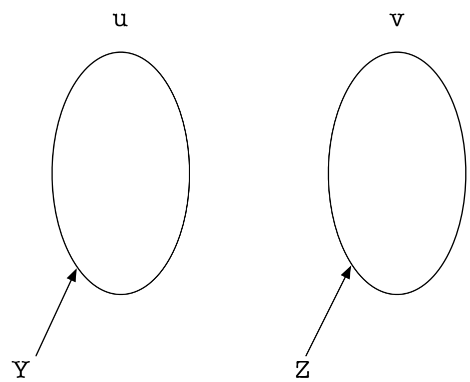

Figure 1: Variables in dual correspond to vertices of the two partitions

Hence the dual can be formulated as

$$\min \sum Y_u + \sum Z_v$$
$$Y_p + Z_q \geq 1 \text{ , } Y_p, Z_q \geq 1 \text{ for all edges (p,q)}$$

It can be clearly seen that the dual corresponds to finding the minimum vertex cover of the graph.

DEFINITION 1 *A vertex cover of an undirected graph is a subset of vertices of the graph which contains at least one of the two endpoints of each edge.*

Next we see the how to find a minimum vertex cover for a graph using a maximum mathcing for that graph.

3 Minimum Vertex Cover from Maximum Matching

If V denotes a vertex cover and M a maximum matching for the same graph, because any vertex cover has to include at least one endpoint of each matched edge, so

$$|V| \geq |M|$$

Also it can be shown that *size of minimum vertex cover for a graph is equal to the size of its maximum matching.*

Now, given a maximum matching how do we find a vertex cover?

Let G(V,E) be a graph and M be a maximum matching for the graph. Let V' be the set of those vertices in M which are at odd distance from unmatched vertices in G wrt M. Then V' is a minimum vertex cover for G

An intuitive way to understand the above is shown in Fig 2.

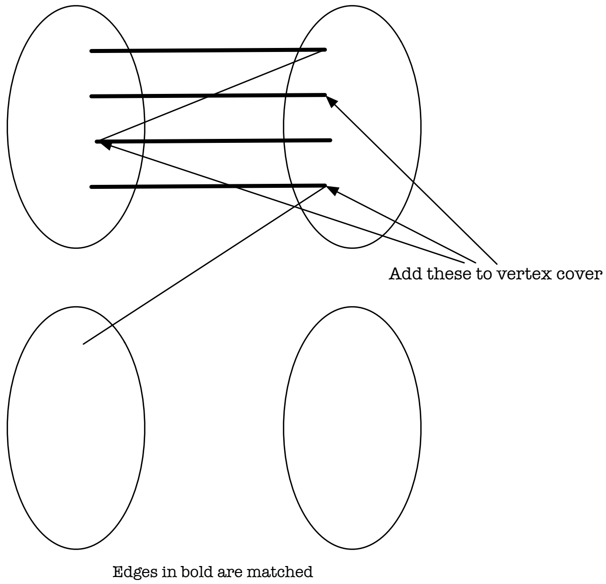

Figure 2: Minimum Vertex Cover from Maximum Matching

In the next lecture, we will look at the case of mathching in bipartite graphs with weights associated with edges.