Lecture 28: Primal-dual algorithm for matching in bipartite graphs (contd.)

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1 Recap

In the previous lecture, we discussed the ILP formulation for *unweighted*[1] bipartite matching. It was observed that the dual for this particular case corresponds to the *minimum vertex cover* for the graph.

In this lecture we will talk about the more general case of matching in a weighted bipartite graph.

2 ILP formulation of Matching in a Weighted Bipartite Graph

The input is a graph with each edge having a positive weight W_{uv}.

Definition 1 *A maximum weighted bipartite matching is defined as a perfect matching where the sum of the values of the edges in the matching have a maximal value.*

As we will later see, the size of such a matching would be n^2. Note that, if the graph is not complete bipartite, missing edges are inserted with value zero.

The variable X_{uv} is defined as,

$$X_{uv} = \begin{cases} 0 & \text{if edge (u,v) belongs to the matching,} \\ 1 & \text{if edge (u,v) does not belong to the matching} \end{cases}$$

Again, as we have added zero-weighted edges to *complete* the graph, we assert that the summations $\forall u \sum X_{uw}$ and $\forall v \sum X_{pv}$ be *exactly* equal to 1.

Therefore, the ILP (*primal*) for this problem is:

$$\max \sum W_{uv} X_{uv}$$

$$\forall u \sum X_{uw} = 1$$

$$\forall v \sum X_{pv} = 1$$

$$\forall u, v \; X_{uv} \geq 0, X_{uv} \text{ is integral}$$

The resultant dual is:

$$\min \sum Y_u + \sum Z_v$$
$$Y_u + Z_v \geq W_{uv}$$

[1]Unweighted: The weight of each edge in the graph is 1
[2]where n is the number of vertices in the graph

3 Algorithm to find Maximum Weighted Bipartite Matching

- **Step 1 - Initialization:** For all vertices v, initialize Y_v and Z_v to the highest weight edge incident on it.
- **Step 2 - Iteration:** Now, locate all edges *(u,v)* for which the equality

  $$Y_u + Z_v = W_{uv}$$

  holds.
  Let E represent the set of such edges. Then the following dual can be formulated for edges in E,

  for each edge (u,v) in E
  $$\min \sum Y'_u + \sum Z'_v$$
  $$Y'_u + Z'_v \geq 0$$

  This version of the dual is unweighted, and hence easier to solve. In the optimum solution, we have $\sum Y'_u + \sum Z'_v = 0$.

  In the next lecture we complete this algorithm.