CS 435 : Linear Optimization — Fall 2006

Lecture 29: Primal-dual algorithm for matching in bipartite graphs (contd.)

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Problem

In weighted matching problem a number $w_{uv} \geq 0$ is associated with each edge of the graph, called the *weight* of that edge and we are suppposed to find a matching with the largest possible sum of weights.

Let the variable X_{uv} be defined as follows:

$$X_{uv} = \begin{cases} 0 & \text{if edge (u,v) is not matched,} \\ 1 & \text{if edge (u,v) is matched} \end{cases}$$

Our task is to maximize $\sum w_{uv} X_{uv}$.

The Primal for matching in weighted bipartite graph is :

$$\begin{aligned} & max \sum W_{uv} X_{uv} \\ & \sum X_{uv} = 1 \\ & X_{uv} \geq 0 \end{aligned} \tag{1}$$

The Dual for the above is :

$$\begin{aligned} & min \sum_u Y_u + \sum_v Z_v \\ & \forall\ edge\ \{u,v\} \quad Y_u + Z_v \geq W_{uv} \end{aligned} \tag{2}$$

Algorithm

1. We start with any feasible dual solution, which may not necessarily be the optimal solition.

2. Let E' be the set of those edges for which the inequalities are tight. For these edges the dual is rewritten as :

$$\begin{aligned} & min \sum Y'_u + \sum Z'_v \\ & \forall\ edge\ \{u,v\} \ \in \ E', \quad Y'_u + Z'_v \geq 0 \end{aligned} \tag{3}$$

   Clearly this problem is unbound, hence to make it bounded, the following equations need to added to the system :

$$\begin{aligned} -1 \ \leq \ Y'_u \ \leq \ 1 \\ -1 \ \leq \ Z'_v \ \leq \ 1 \end{aligned} \tag{4}$$

3. The optimal solution to the above system is the point at which the costs of the primal feasible and dual feasible coincide.

   Treating the above equation as primal, B = 0. Hence, the dual of the above dual thus is:

$$\begin{aligned} &max\ (0) \\ &\forall\ edge\ \{u,v\}\ \in\ E',\ \sum X_{uv} = 1 \\ &X_{uv}\ \geq\ 0 \end{aligned} \tag{5}$$

   This equation will give a matching of size n in E' by the following algorithm repeated recursively :

   - If the current matching is of size n, then the dual is optimal and we have found the matching of maximum size.
   - Else, if the Dual is not optimal, then the matching in E' is less than size n.
   - Let the size of the Dual be t. Then, in the vertex cover comprising of the t vertices, we increase the value of all the vertices by δ. And for all the remaining vertices (not in the vertex cover), we decrease the size by δ. Since there are 2n vertices and t<n, the total cost will decrease.