Lecture 29: Primal-dual algorithm for matching in bipartite graphs (contd.)

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1. Primal

$$\begin{aligned} & max \sum W_w X_{uv} \\ & \sum X_{uv} = 1 \\ & X_{uv} \geq 0 \end{aligned} \tag{1}$$

2. Dual

$$\begin{aligned} & min \sum_u Y_u + \sum_v Z_v \\ & \forall \; edge \; \{u,v\} \quad Y_u + Z_v \geq W_{uv} \end{aligned} \tag{2}$$

Algorithm

1. Start with any feasible dual solution.
2. Let E' be the set of edges for which the inequalities are held.

$$\begin{aligned} & min \sum Y'_u + \sum Z'_v \\ & \forall \; edge \; \{u,v\} \; \in \; E', \quad Y'_u + Z'_v \geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned} -1 \; \leq \; Y'_u \; \leq \; 1 \\ -1 \; \leq \; Z'_v \; \leq \; 1 \end{aligned} \tag{4}$$

   Equation 4 is just to make sure that the LP is bounded.

3. If we find a primal feasible and dual feasible with the same cost then we are done. It is the optimal solution.

   The dual of equation 3 :

$$\begin{aligned} & max \; 0 \\ & \forall \{u,v\} \; \in \; E', \; \sum X_{uv} = 1 \\ & X_{uv} \; \geq \; 0 \end{aligned} \tag{5}$$

   Equation 5 must be matching of the size n in E'.

4. If we find a matching of maximum size then we are done.

5. If we know that the Dual is not optimal, then it means that matching in E' is not of size n max and we need to improve the solution.

6. What if the matching is not of size n in E'? How to improve?

   Suppose the matching is of size t then there is a vertex cover of size t. In this vertex cover, for all the vertices increase the value of the vertex by δ. For vertices not in the vertex cover, decrease the value by δ.

   Since there are total 2n vertices and t $<$ n so the total cost will decrease.

   We increase with best δ we can, such that one more edge tighten and another edge comes in E'.