Lecture 3: Linear algebra I

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1 Vector Space

A *vector space* is defined as a set of vectors $\mathbf{V}$ and the real numbers $\mathbf{R}$ (called *scalars*) with the following operations defined:

- **Vector Addition:** $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$, represented as $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathbf{V}$.
- **Scalar Multiplication:** $\mathbf{R} \times \mathbf{V} \rightarrow \mathbf{V}$, represented as $a.\mathbf{u}$, where $a \in \mathbf{R}$ and $\mathbf{u} \in \mathbf{V}$.

Following are the properties of a vector space.

- **Abelian Group laws:**
  1. **Associativity:** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  2. **Identity:** $\exists$ a zero vector $\overline{\mathbf{0}}$ which is the group identity element, i.e. $\overline{\mathbf{0}} + \mathbf{u} = \mathbf{u}$
  3. **Inverse:** $\forall \mathbf{u} \in \mathbf{V}$, there exists the additive inverse $-\mathbf{u}$ s.t. $\mathbf{u} + (-\mathbf{u}) = \overline{\mathbf{0}}$
  4. **Commutativity:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Scalar multiplication laws:**
  1. **Multiplication by** 0**:** $0.\mathbf{u} = \overline{\mathbf{0}}$
  2. **Multiplication by -1:** $(-1).\mathbf{u} = -\mathbf{u}$
  3. **Identity multiplication:** $1.\mathbf{u} = \mathbf{u}$
  4. **Distributivity of vector sum:** $a.(\mathbf{u} + \mathbf{v}) = a.\mathbf{u} + a.\mathbf{v}$, where $a \in \mathbf{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbf{V}$
  5. **Distributivity of scalar sum:** $(a + b).\mathbf{u} = a.\mathbf{u} + b.\mathbf{u}$
  6. **Associativity of scalar multiplication:** $a.(b.\mathbf{u}) = (ab).\mathbf{u}$

2 Subspace

$\mathbf{U} \subseteq \mathbf{V}$ is a *subspace* of $\mathbf{V}$ if $\mathbf{U}$ itself is a vector space, i.e. for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}$ and $\alpha \in \mathbf{R}$, $\mathbf{u}_1 + \mathbf{u}_2 \in \mathbf{U}$ and $\alpha.\mathbf{u}_1 \in \mathbf{U}$. For example, if $\mathbf{u} \in \mathbf{V}$, then $\mathbf{U} = \{\alpha.\mathbf{u} \ \forall \alpha \in \mathbf{R}\}$ is a subspace.
For $\alpha = -1$, $-\mathbf{u}_1 \in \mathbf{U}$ whenever $\mathbf{u}_1 \in \mathbf{U}$. Hence, $-\mathbf{u}_1 + \mathbf{u}_1 \in \mathbf{U}$. Therefore, $\overline{\mathbf{0}}$ is always a member of any subspace.

Example 1 In a 2-dimensional space, any line passing thru the origin is a subspace. If there is any vector in $\mathbf{U}$ that does not lie on this line, then $\mathbf{U}$ has to be the entire plane.

3 Linear Dependence, Independence and basis

DEFINITION 1 *Vectors* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *are linearly dependent if there exist* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$*, not all zero, such that*

$$\sum_{i=1}^{n} \alpha_i . \boldsymbol{v}_i = \overline{\boldsymbol{0}}$$

DEFINITION 2 *Vectors* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *are linearly independent if they are not linearly dependent, i.e. for* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$

$$\sum_{i=1}^{n} \alpha_i . \boldsymbol{v}_i = \overline{\boldsymbol{0}} \quad \Rightarrow \quad \alpha_i = 0, \ \forall i$$

Basis of a vector space is defined in terms of linear dependence as follows:

DEFINITION 3 *Vectors* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *form the basis of a vector space* $\mathbf{V}$ *iff:*

1. *They are linearly independent.*
2. *Every other vector* $\boldsymbol{w}$ *which belongs to* $\mathbf{V}$ *can be written as*

$$\boldsymbol{w} = \sum_{i=1}^{n} \beta_i . \boldsymbol{v}_i$$

Alternatively, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *form the basis of vector space* $\mathbf{V}$ *if on adding other vector* $\boldsymbol{w} \in \mathbf{V}$ *to this set, the set becomes linearly dependent.*

There can be multiple basis for the same vector space, but all of them will have the same size. Therefore, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis, and $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is also a basis, then $m = n$. The number of vectors in the basis is called the *dimension* of the vector space.