Lecture 31: Network Flows

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In this lecture we introduced network flows and primal-dual algorithm for same.

1 Definition

A network flow is an assignment of flow to the edges of a directed graph with each edge havinf a capacity, such that the amount of flow along an edge doesnot exceed its capacity.

The Network problem can also be seen as one wants to send as many trucks from s to t. Trucks can't stop at any node. What is the number of trucks on each of these roads?

Hence, we have to solve the following problem.

2 Input

- Directed Graph G(V,E) with nodes V and edges E.
- Positive integer capacity on each edge c_{uv}.
- Two special vertices s(source) and t(sink).

NOTE: Parallel edges are disallowed for convinience. They can be easily converted into non-parallel edges by adding one vertex between them.

3 Output

For each edge uv, flow at that edge f_{uv} such that

$$\sum_u f_{su} \text{ is maximum}$$

A network flow has following three properties for all nodes u and v :

$$\forall u, v \ f_{uv} \leq C_{uv}$$

i.e. the flow along an edge cannot exceed its capacity.

$$\forall u, v \ f_{uv} + f_{vu} = \ 0$$

The net flow from u to v must be opposite of the net flow from v to u.

$$\forall u \neq s, t \ \sum_v f_{uv} = 0$$

i.e. The net flow to a node is zero.

4 Algorithm

Here, it is better to solve the primal itself than the corresponding dual.

4.1 Initialization

Initialize :

$$\forall u, v \ f_{uv} = 0$$

Now, we need a code that can run recursively, resulting in improvement of the above solution till we find the optimal. The Primal-Dual algorithm will ne helpful in finding that.

4.2 Recursion

New network is :

$$max \sum_u f'_{su}$$

$$\text{s.t. } \forall u, v \text{ where } (\ f_{uv} = C_{uv} \ ) \ f'_{uv} \leq \ 0 \quad (1)$$

$$\forall u, v \ f'_{uv} + f'_{vu} = \ 0 \quad (2)$$

$$\forall u \neq s, t \ \sum_v f'_{uv} = 0 \quad (3)$$

Now the relation between original problem and this is :

$$\forall u, v \ f^{old}_{uv} + \epsilon f'_{uv} = f^{new}_{uv}$$

Our aim is to keep raising ϵ until we reach an equality of the form $f^{new}_{uv} = C_{uv}$

NOTE: If $f_{uv} \leq 1$, then the problem will be just to find is there a path from s to t.

4.3 Example

Let us now consider the case given in Fig. 1(a). Let all capacities be 1.
Now let us assume that current flow is ABCD (i.e. $f_{AB} = f_{BC} = f_{CD} = 1$). Then the new

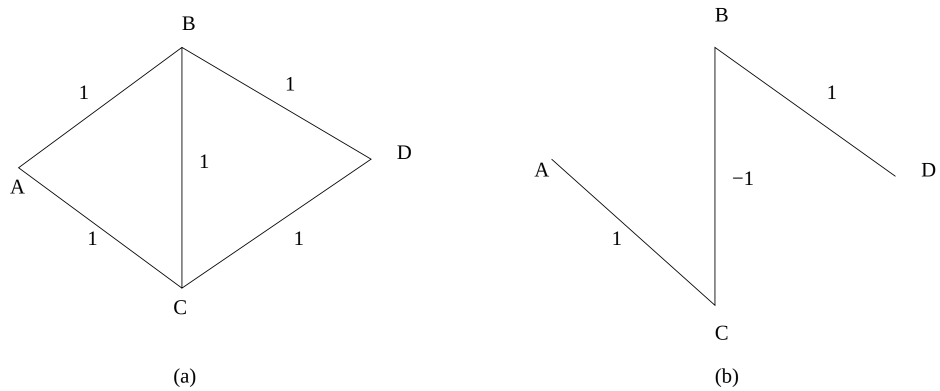

Figure 1: Network with all capacities equal to 1

network is Fig. 1(b). Here $-1 \leq 1$, hence the edge BC will be there in new network.

5 Next class

We will continue with this in next class and look at the algorithm more closely.