CS 435 : Linear Optimization — Fall 2006

Lecture 4: Linear algebra : Basis, Dimension

Lecturer: *Sundar Vishwanathan* — Scribe: *Bhaskara Aditya*
Computer Science & Engineering — Indian Institute of Technology, Bombay

1 Basis (contd.)

We first give the proof for the result stated in the previous lecture – if two sets of vectors S and T form a basis for a vector space V, their sizes are the same (i.e., $|S| = |T|$). We recall what is meant by saying a set X of vectors is a basis for a space V:

1. The vectors in X are linearly independent (i.e., there's no nontrivial combination of them which is zero).
2. Any vector in V can be expressed as a linear combination of vectors in X.

We will prove the result by contradiction, by assuming $|S|$ is strictly smaller than $|T|$, and then obtaining a set $S' \subset T$ which is also a basis, with the property $|S'| = |S|$. This is clearly a contradiction, because $|S'| = |S| < |T|$ and $S' \subset T$, so the vectors in $T \setminus S'$ can be expressed in terms of those in S', contradicting the fact that T is a basis (in particular, (1) above).

We start with a lemma.

Lemma 1 *Suppose $S = \{v_1, \dots, v_n\}$ and let $x = \sum_{i=1}^{n} \alpha_i v_i$ with $\alpha_1 \neq 0$. Let $S' = \{x, v_2, \dots, v_n\}$. Then, span$(S)$ = span(S').*

Proof: The proof in both directions is easy. The crucial thing is to observe that since $\alpha_1 \neq 0$, we can write v_1 as a combination of $x, v_2, \dots, v_n$. Thus, any vector in *span*(S) is also in *span*(S'). For the other direction, since x is a combination of v_i's, every vector in *span*(S') is also in *span*(S). This proves the equality. □

We now proceed with the proof of the theorem.

Theorem 1 *Suppose $S = \{u_1, u_2, \dots, u_m\}$ and $T = \{v_1, v_2, \dots, v_n\}$ be two sets of vectors such that each is a basis for the vector space V. Then $|S| = |T|$.*

Proof: The proof will follow the outline above. Suppose without loss of generality, that $m < n$. Starting with the set $S_0 = S$, we do the following: replace one of the vectors of S_0 by a vector from T such that the new set, say S_1, still spans the entire V. Also, if v_i was the vector in T which was added to S_0, we set $T_1 = T \setminus \{v_i\}$.

We now repeat this step m times. Further, we ensure that at each step, the element removed from S_i is one of the u_j's (not the v_j's that have been added). We now show that such an operation is indeed always possible. More precisely, assume that we have two sets S_i and T_i (with $i < m$). We show that it is always possible to obtain a set S_{i+1} such that:

1. S_{i+1} is obtained from S_i by removing one of the u_j's from S_i and adding one of the v_j's (which is from T_i) (call it x) to it.

2. The span of S_{i+1} is the same as the span of S_i.

Further, we set $T_{i+1} = T_i \setminus \{x\}$. We prove this as follows.

Note that since initially we have that $S_0 = S$ is a basis, and at every step so far the span was preserved, we can assume S_i spans the entire V. We may also assume that we have re-numbered the u_i's and v_i's such that $S_i = \{u_{i+1}, \ldots, u_m, v_1, \ldots, v_i\}$ and $T_i = \{v_{i+1}, \ldots, v_n\}$.[1] Since S_i spans the whole of V, we have

$$v_{i+1} = \sum_{j=i+1}^{m} \alpha_j u_j + \sum_{j=1}^{i} \beta_j v_j \tag{1}$$

Now, at least one of α_j's must be non-zero, else we would have a non-trivial combination of v_i's as zero, which is not possible since T is a basis. So assume without loss of generality α_{i+1} is non-zero. Then, by the lemma above, replacing u_{i+1} by v_{i+1}, we still get a basis. This is the required S_{i+1}.

Thus, if we repeat this process m times, the resulting set, S_m will have just v_j's and they span the whole of V. This is a contradiction, as we have seen in the outline above. □

So we have proved that given a vector space V, any basis for it will have the same size (assuming there exists a finite basis). Thus, this number is a property of V alone, and it is called the **dimension** of V. The above result proves that it is well-defined.

2 Solutions to $Ax = \mathbf{0}$

We now want to look at the set of all solutions to the system $Ax = \mathbf{0}$, where A is an $m \times n$ matrix and x is a vector in $\mathbb{R}^n$. From now on, denote $\mathcal{S} = \{x \ : \ Ax = \mathbf{0}\}$. Note that $\mathcal{S}$ is a subspace of $\mathbb{R}^n$. This is because if $x \in \mathcal{S}$ then $A(\alpha x) = \alpha(Ax) = \mathbf{0}$, so $\alpha x \in \mathcal{S}$. Similarly, we can verify that $x_1, x_2 \in \mathcal{S} \Rightarrow x_1 + x_2 \in \mathcal{S}$.

Having proved that $\mathcal{S}$ is a subspace, we ask the natural question – what is it's dimension? We give an answer to this problem in terms of the matrix A. We will, in fact, prove the following in the coming lectures.

THEOREM 2 1 *Suppose k is the number of linearly independent columns in the matrix A. Then, $dim(\mathcal{S}) = n - k$.*

We will also prove a 'row version' of this theorem.

THEOREM 3 1 *Suppose k is the number of linearly independent rows in the matrix A. Then, $dim(\mathcal{S}) = n - k$.*

This gives, as an interesting and non-trivial corollary, that the number of linearly independent rows in a matrix is equal to the number of linearly independent columns. This is interesting, because it is not obvious at all, at first sight. This number is defined to be the **rank** of the matrix A.

We will end with some ideas relating to the proof of Theorem 2. One observation is the fact that $x = (x_1\ x_2\ \ldots\ x_n)^T$ is in $\mathcal{S}$ iff $\sum_{i=1}^{n} x_i A^{(i)} = \mathbf{0}$ where $A^{(i)}$ is the ith column of the matrix A. This is easy to see by writing out the above summation.

[1] If $i = 0$, $S_0 = S$, $T_0 = T$, and we interpret summations of the form $(\sum_{j=1}^{i} \ldots)$ as zero.