Lecture 5: Linear algebra : column space of matrix A, solution space of Ax=0, relationship between them

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Last Lecture : Given vectors v_1,v_2,v_3,...,v_n the space spanned by them is $\sum_{i=1}^{n} \alpha_i v_i$. Thus $\sum_{i=1}^{n} \alpha_i v_i$ is a subspace.

Consider the space of vectors $\{x : Ax = \overline{0}\}$.

Looking by the column perspective

$$Ax = b \Rightarrow A^1 x_1 + A^2 x_2 + \ldots + A^n x_n = b$$

here, A^i is the i^{th} column of A and x_i is the i^{th} component of vector x

which means b is in the column space of A. Hence, in the given situation

$$A^1 x_1 + A^2 x_2 + \ldots + A^n x_n = \overline{0}$$

Assume : $A^1, A^2, \ldots, A^k$ are a basis for the space spanned by $A^1, A^2, \ldots, A^n$, $k \leq n$.

which implies,

$$\begin{aligned} A^{k+1} &= \textstyle\sum_{j=1}^{k} \alpha_j^{k+1} A^j \\ A^{k+2} &= \textstyle\sum_{j=1}^{k} \alpha_j^{k+2} A^j \\ &\ldots \\ A^{n} &= \textstyle\sum_{j=1}^{k} \alpha_j^{n} A^j \end{aligned}$$

hence, dimension of this space is atleast $n - k$

$$dim\{x : Ax = \overline{0}\} \geq n - k$$

now, to prove that it is EXACTLY $n - k$.

Proof: Take an x_0 such that $Ax_0 = \overline{0}$. Also let U_i be such that,

U_{k+1} is

$$\begin{pmatrix} \alpha_1^{k+1} \\ \alpha_2^{k+2} \\ \vdots \\ \alpha_k^{k+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

U_{k+2} is

$$\begin{pmatrix} \alpha_1^{k+2} \\ \alpha_2^{k+2} \\ \vdots \\ \alpha_k^{k+2} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now consider the vectors x and $x\prime$ such that,

$$x = [x_1 \dots x_n]^T$$
$$x\prime = x - \{x_{k+1}U_{k+1} + \dots + x_nU_n\}$$

but

$$Ax = \overline{0} \qquad \text{and}$$
$$A\{x_{k+1}U_{k+1} + \dots + x_nU_n\} = \overline{0}$$

because given that, x and U_i are from the null space of A. Therefore

$$Ax\prime = \overline{0}$$

note that because of way in which $x\prime$ is defined, its last $n-k$ components are zero. Hence above equation means that the combination of first k columns is also zero. Since the first k columns are linearly independent, therefore, the linear combination is *trivial.*Hence,

$$x_1\prime = x_2\prime = \dots = x_n\prime$$

therefore, the dimension of $\{x : Ax_0 = \overline{0}\}$ is EXACTLY $n-k$.

In the next lecture, we will analyse the row perspective.