CS 435 : Linear Optimization | Fall 2006

Lecture 7: Linear Algebra

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1 Convex Sets

Let us start by defining a *convex combination.*

Definition 1 *Given a set of vectors v_i, any vector of the form*

$$v = \sum_{i=1}^{n} \lambda_i v_i, \quad 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{n} \lambda_i = 1$$

is called a convex combination of the v_i's.

Now, using the above definition, a *Convex Set* can be defined as:

Definition 2 *A set S of vectors(points) is called convex if all convex combinations of vectors in S are in S.*

Another way of looking at the definition is:

Definition 3 *If for any two points in S, the line segment joining them is in S, then the set S is said to be convex.*

Figure 1: Example of a convex and a non-convex polygon

We can see in the figure that any line segment joining 2 points in the convex region will lie in the region itself. Thus the region is convex. This is not true with the other non-convex figure. We can see in the figure a line segment which joins 2 points in the polygon is not completely situated in the polygon and hence the polygon is not convex.

Theorem 1 *If S_1 and S_2 are two convex sets, then $S_1 \cap S_2$ is a convex set.*

Proof: Let $x_1, x_2 \in S_1 \cap S_2$. Now since x_1 and x_2 belong to S_1 (which is convex), any convex combination of them lies in S_1. Similarly we can say that this convex combination of x_1 and x_2 lies in S_2. Thus the convex combination lies in $S_1 \cap S_2$. Thus $S_1 \cap S_2$ is convex. □

Theorem 2 *$\{x : Ax \leq b\}$ is a convex set.*

Proof: Let x_1 and x_2 be in the set. Then

$$Ax_1 \leq b \text{ and,}$$

$$Ax_2 \leq b$$

Consider λ_1 and λ_2 (> 0) such that $\lambda_1 + \lambda_2 = 1$
Then,

$$\lambda_1(Ax_1) + \lambda_2(Ax_2) \leq b(\lambda_1 + \lambda_2)$$

$$\Rightarrow \lambda_1(Ax_1) + \lambda_2(Ax_2) \leq b$$

$$\Rightarrow A(\lambda_1 x_1 + \lambda_1 x_2) \leq b$$

Thus, $\{x : Ax \leq b\}$ is convex. □

An alternate proof could be as follows. Let us look at $A_1 x \leq b_1$. All the points satisfying this ineqality lie on one side of the hyper-plane $A_1 x = b$. Thus the set formed by these points is convex (it is easy to check). Similarly the sets of solutions to the inequalities $A_2 x \leq b_2$, $A_3 x \leq b_3 \ldots (A = [A_1, A_2, \ldots])$ are also convex.

Thus, A can be seen as an intersection of Convex regions. And from Theorm 1 we can argue that since set A is an intersection of convex regions, the set A is convex.

2 Maximize $c^T x$

How do we maximize $c^T x$ over the set of all x satisfying $Ax \leq b$?

First, consider maximizing over the region $x^T x \leq 1$. This is a set such that all points are at a distance less than or equal to 1 from the origin. We can see that this is a convex set.

We know that $c^T x$ increases in the direction of c. Thus we start moving in the direction of c. The last point where $c^T x$ touches the sphere is the point of maxima. It can be easily seen that at this point, $c^T x$ is a tangent to the sphere $x^T x$.

Now consider any convex polygon on a 2-d plane. To maximize $c^T x$ in 2-d keep moving along c. The last point where $c^T x$ touches the polygon will the point of maxima. It can be observed that this point will be a boundary point.

Extending this argument to a n-dimensional plane, it seems that $c^T x$ will attain its maximum value at the boundary points of the region $Ax \leq b$.

A local maximum of a function f can be defined as follows.

Definition 4 *If there exists a small neighbourhood N of x_0 where $f(x_0) \geq f(x) \forall x \in N$, then x_0 is said to be a point of local maxima.*

Theorem 3 *If f is linear and f convex, then a local maximum is a global maximum.*

Proof: Let x_0 be a local maximum and y be a global maximum.
Consider a point $P = (1 - \epsilon)x_0 + \epsilon y$. If $\epsilon \to 0$, then P lies in any small neighbourhood of x_0 (in particular, in the neighbourhood where x_0 is the point of local maximum).
Now, consider $f((1 - \epsilon)x_0 + \epsilon y)$. Since f is linear, this can be written as

$$(1 - \epsilon)f(x_0) + \epsilon f(y)$$

$$= f(x_0) + \epsilon(f(y) - f(x_0))$$

We can now observe that at point P, which is in the neighbourhood of x_0, the value of the function $f((1 - \epsilon)x_0 + \epsilon y) \geq f(x_0)$. Thus $f(x_0)$ can be maximum only if $f(x_0) = f(y)$.
Thus, we can see that a local maximum is the same as a global maximum. $\square$