Lecture 8: Extreme points

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In this lecture, we define *extreme points* in geometry and try to give an linear algebra perspective for the same.

Definition 1 *Given points $p_1, p_2, p_3, \ldots, p_n$, the convex hull is the smallest convex set containing these points.*

To simplify the discussion, we make the following assumptions on the nature of $Ax \leq \mathbf{b}$ which hold for the rest of the course unless otherwise stated.

1 Assumptions on the nature of the convex set $Ax \leq \mathbf{b}$

- *Assumption 1:* $Ax \leq \mathbf{b}$ is bounded. No single coordinate of x satisfying $Ax \leq \mathbf{b}$ can increase or decrease without bound.
- *Assumption 2:* $Ax \leq \mathbf{b}$ has no degeneracies. This is equivalent to saying that not more than n hyperplanes pass through a point in an n-dimensional space. This assumption makes sense because small perturbations of the hyperplanes can remove the degeneracies.
- *Assumption 3:* $Ax \leq \mathbf{b}$ should be *full dimensional.* This is equivalent to saying that one should be able to place a n-dimensional sphere, however small, in the region defined by $Ax \leq \mathbf{b}$. For two dimensions, the convex set should have area, and for three dimensions, the convex set should have volume. In n dimensions, the convex set should have an n-dimensional volume.

2 Extreme Points and its Algebraic Interpretation

Definition 2 *An extreme point is a point in a convex set that cannot be represented as a convex combination of any two other distinct points in the convex set.*

Thus, an extreme point does not lie on the segment between two other distinct points of the convex set. Intuitively, all segments with the extreme point in the interior "stick out" of the convex set.

Theorem 1 *Given a convex set $Ax \leq \mathbf{b}$ in n-dimensional space satisfying the assumptions of Sec. 1, the extreme points of a convex set are precisely those points in $Ax \leq \mathbf{b}$ which can be expressed as an intersection of n linearly independent hyperplanes out of the set of hyperplanes that define the convex set $Ax \leq \mathbf{b}$.*

PROOF:
Part 1: ($\Rightarrow$) Any point which can be expressed as an intersection of n linearly independent hyperplanes out of the set of hyperplanes that define the convex set $Ax \leq \mathbf{b}$ is an extreme point.

Let x_0 be any such point. We split $Ax_0 \leq \mathbf{b}$ into two parts

$$A^{'}x_0 = \mathbf{b}^{'} \quad (1)$$
$$A^{''}x_0 < \mathbf{b}^{''} \quad (2)$$

(We effectively move the inequalities in $Ax_0 \leq \mathbf{b}$ that are equalities into the first set $A^{'}x_0 = \mathbf{b}^{'}$)

We notice that since x_0 is a point on the intersection of n linearly independent hyperplanes in the defining set of hyperplanes, $A^{'}$ has n linearly independent rows.

Let there be x_1 and x_2 in the given convex set such that $x_0 = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$. We then have

$$\lambda A^{'}x_1 + (1-\lambda)A^{'}x_2 = \mathbf{b}^{'} \quad (3)$$

But since x_1 and x_2 are part of the convex set, $A^{'}x_1 \leq \mathbf{b}^{'}$ and $A^{'}x_2 \leq \mathbf{b}^{'}$.

Neither of these inequalities can be strict (or else Eq. 1 will not hold) and therefore $A^{'}x_1 = \mathbf{b}^{'}$ and $A^{'}x_2 = \mathbf{b}^{'}$.

Now, we are given that x_0 is a solution of $A^{'}x = \mathbf{b}^{'}$, which is nothing but the intersection of n linearly independent hyperplanes. Thus, we have $x_0 = x_1 = x_2$.

Thus x_0 cannot be expressed as a convex combination of two other distinct points in the convex set and is therefore an extreme point.

Part 2: ($\Leftarrow$) Let x_0 be an extreme point. If we split $Ax_0 \leq \mathbf{b}$ into two parts

$$A^{'}x_0 = \mathbf{b}^{'} \quad (4)$$
$$A^{''}x_0 < \mathbf{b}^{''} \quad (5)$$

then A' has n linearly independent rows.

See next lecture for a proof.

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