Lecture 9: Convex hull of extreme points

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In this lecture, we complete the proof of the thoerem on extreme points mentioned in the previous lecture and then begin the proof of the thoerem on convex hull of the extreme points of $\{x : Ax \leq \mathbf{b}\}$.

Proof(contd): Here we prove the converse relation that is every extreme point of $\{x : Ax \leq \mathbf{b}\}$ can be expressed as an intersection of n linearly independent hyperplanes
We prove this by showing that if a point cannot be expressed as an intersection of n linearly independent hyperplanes then we can express it as a convex combination of two points in the set.
Let x_0 be a point. We split $Ax_0 \leq \mathbf{b}$ into two parts

$$A^{'}x_0 = \mathbf{b}^{'} \quad (1)$$
$$A^{''}x_0 < \mathbf{b}^{''} \quad (2)$$

Since $A^{''}x_0$ is strictly less than $\mathbf{b}^{''}$,we can draw a small enough sphere around x_0 such that every point x within the sphere satifies $A^{''}x_0 < \mathbf{b}^{''}$. This means that $\exists$ a (= radius of this sphere) such that for which all vectors $\bar{\epsilon}$ which have magnitude($|\bar{\epsilon}|$) $\leq a$ we have

$$A^{''}(x_0 + \bar{\epsilon}) < \mathbf{b}^{''} \quad (3)$$

Now assume that $A^{'}$ does not have n linearly independent vectors. Then $A^{'}x = 0$ will have a non zero solution. Let x_1 be a non zero solution of $A^{'}x = 0$. Then it is easy to observe $\forall \lambda \in R$

$$A^{'}(x_0 + \lambda x_1) = \mathbf{b}^{'} \quad (4)$$
$$A^{'}(x_0 - \lambda x_1) = \mathbf{b}^{'} \quad (5)$$

By making λ small we can make magnitude of λx_1 to become less than a and so by Eq. 1

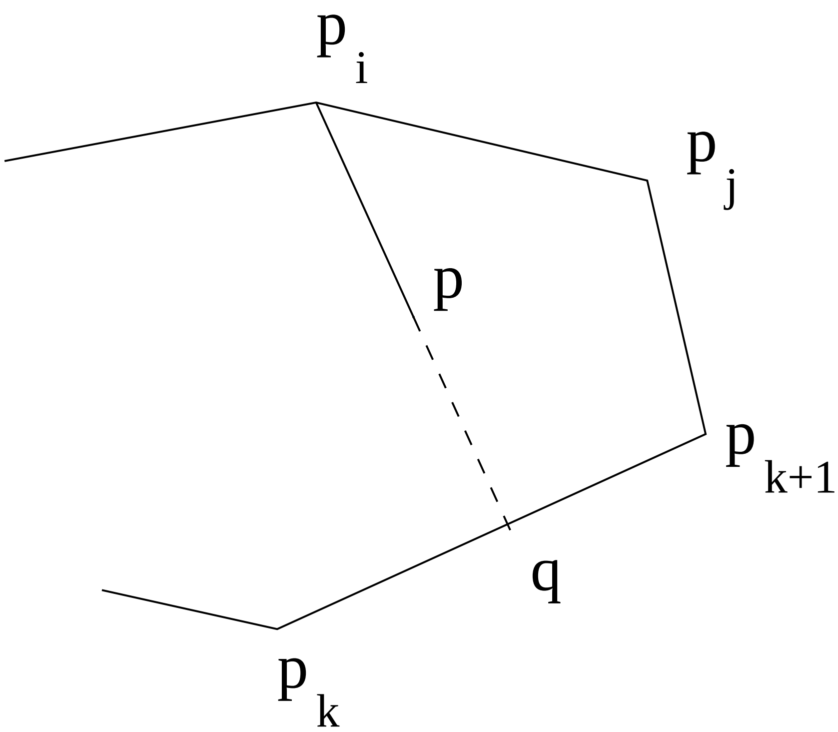
$$A^{''}(x_0 + \lambda x_1) < \mathbf{b}^{''} \quad (6)$$
$$A^{''}(x_0 - \lambda x_1) < \mathbf{b}^{''} \quad (7)$$

So $x_0 + \lambda x_1$ and $x_0 - \lambda x_1$ are points in $\{x : Ax \leq \mathbf{b}\}$ and also $x_0 = (1/2)(x_0 + \lambda x_1) + (1/2)(x_0 - \lambda x_1)$. Thus we have expressed x_0 as a convex combination of two points in the set. This is a contradiction !. Hence we are proving the theorem.

1 Convex hull of the extreme points

Definition 1 *A convex hull of finite points $p_1, p_2, \ldots p_n$ is the set of all points p which can be written as convex combination of $p_1, p_2, \ldots p_n$.*

We have the following theorem



THEOREM 1 *Let $p_1, p_2, \ldots p_n$ be extreme points of $x : Ax \leq \mathbf{b}$. Then $x : Ax \leq \mathbf{b}$ is the convex hull of the points $p_1, p_2, \ldots p_n$*

PROOF:
It is easy to observe that any convex combination of $p_1, \ldots p_n$ belongs to the set since $p_1, \ldots p_n$ belongs to $x : Ax \leq \mathbf{b}$. So

$$\text{Convex hull of } p_1, \ldots p_n \subset \{x : Ax \leq \mathbf{b}\} \tag{8}$$

What remains to show is that any x_0 such that $Ax_0 \leq \mathbf{b}$ can be written as

$$x_0 = \Sigma\lambda_i p_i \text{ where } \Sigma\lambda_i = 1 \text{ , } 0 \leq \lambda_i \leq 1 \tag{9}$$

We show this by induction on dimensions.
Base case : Consider the 2-D space and a convex set $\{x : Ax \leq \mathbf{b}\}$ in it. Let p be any point inside the set. Now we take the extreme point p_i and join it to p. Next we extend this line untill it touches one of the bounding segments ($p_k p_{k+1}$ in this case) at some point (say q)
Since q lies on the segment joining p_k and $p_k + 1$ it can be expressed as a convex combination of p_k and $p_k + 1$. Therefore,

$$q = \lambda_1 p_k + \lambda_2 p_k + 1 \text{ where } \lambda_1 + \lambda_2 = 1. \tag{10}$$

Also p lies on the segment joining q and p_i. So p can expressed as a convex combination of q and p_i. Therefore,

$$p = \lambda_3 p_i + \lambda_4 q \text{ where } \lambda_3 + \lambda_4 = 1. \tag{11}$$

Combining the above equations we get

$$p = \lambda_3 p_i + \lambda_4 \lambda_1 p_k + \lambda_4 \lambda_2 p_k + 1 \tag{12}$$

Now $\lambda_3 + \lambda_4\lambda_1 + \lambda_4\lambda_2 = 1$. Thus we have expressed p as a convex combination of the extreme points. Hence the result is true in 2-D.

In next class.

$\square$