1. A symmetric chain decomposition of $\mathcal{B}_{n+1}$ can be constructed from that of $\mathcal{B}_n$. For $n = 1$, $\mathcal{B}_1$ itself is a symmetric chain and forms the required decomposition. Given a symmetric chain decomposition of $\mathcal{B}_n$, for each chain $X_k \subset X_{k+1} \cdots \subset X_{n-k}$, add the chains $X_k \subset X_{k+1} \subset \cdots \subset X_{n-k} \subset (X_{n-k} \cup \{n+1\})$ and the chain $(X_k \cup \{n+1\}) \subset \cdots \subset (X_{n-k-1} \cup \{n+1\})$ if $k < n-k$. It is easy to verify that each of the constructed chains is symmetric and every set in $\mathcal{B}_{n+1}$ is in exactly one chain.

   It is also possible to give an explict description of these chains, that is, an $O(n)$-time algorithm to find the next and the previous element in the chain containing a given subset X, if there is any.

   Since every symmetric chain must contain an element of the middle level(s), the number of chains is at most $\binom{n}{\lfloor n/2 \rfloor}$, hence the maximum antichain has size at most $\binom{n}{\lfloor n/2 \rfloor}$.

2. Suppose A is any antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. By the inequality proved in class, also called the LYM inequality,

$$\sum_{i=1}^{\binom{n}{\lfloor n/2 \rfloor}} \frac{1}{\binom{n}{|A_i|}} \leq 1.$$

   Since $\binom{n}{|A_i|} \leq \binom{n}{\lfloor n/2 \rfloor}$, for this inequality to hold, each term must be equal to $\frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$, that is each A_i has size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. This implies that if n is even, each A_i has size $n/2$. Thus A must be the set of all subsets of size $n/2$. If n is odd, each A_i has size $(n+1)/2$ or $(n-1)/2$. Suppose there are k sets in A of size $(n+1)/2$ where $0 < k < \binom{n}{(n+1)/2}$. Then the Kruskal-Katona theorem implies that the shadow of these sets has size $> k$, which implies some set in A of size $(n-1)/2$ is contained in the shadow. This contradicts the fact that A is an antichain.

   It is also possible to give a direct proof for the case when n is odd. Consider the bipartite graph formed by subsets of size $(n-1)/2$ and $(n+1)/2$, where two subsets are adjacent iff they differ in excatly one element. This graph is regular with each vertex of degree $(n+1)/2$ and it is connected. This implies that the shadow of any collection of k subsets of size $(n+1)/2$ has size $> k$, if $0 < k < \binom{n}{(n+1)/2}$.

3. The proof is again by induction on n. If $n = 1$, the only possible antichains are $\{\emptyset\}, \{1\}$, and the result is trivially true. Suppose the

statement is true for $\mathcal{B}_{n-1}$. Let A be any antichain in $\mathcal{B}_n$ such that every maximal chain in $\mathcal{B}_n$ contains exactly one subset in A. Suppose A contains subsets of different cardinalities and let X be a subset in A of minimum cardinality. Then $|X| < n$ and we may assume, without loss of generality, that $n \notin X$. Let $A' = A \cap \mathcal{B}_{n-1}$. If there is a maximal chain in $\mathcal{B}_{n-1}$ that does not contain any element of A', then by adding the set $\{1, 2, \ldots, n\}$ to it, we get a maximal chain in $\mathcal{B}_n$ that does not contain any element of A, a contradiction. (Note that $\{1, 2, \ldots, n\} \notin A$ otherwise A cannot contain any other subset.) Therefore every maximal chain in $\mathcal{B}_{n-1}$ contains some element of A'. Therfore by induction, A' consists of all subsets of $\{1, 2, \ldots, n-1\}$ with size k, for some k. Since $X \in A'$, we have $k = |X|$. Let Y be a subset in A of cardinality greater than $|X|$ (this exists by assumption). Then $Y \notin A'$ and $n \in Y$. However, by deleting n and possibly some other elements from Y, we get a subset of size k not containing n that is a subset of Y. This subset is in A', which contradicts the fact that A is an antichain.

4. First show that every graph with at least $2m$ vertices and minimum degree at least m contains a matching of size m. This is also by induction. For $m = 1$, this is trivial. Suppose $u_i v_i$, for $1 \leq i \leq m-1$ is a matching of size $m-1$ in a graph with at least $2m$ vertices and minimum degree at least m. Let A be the remaining vertices. If there is an edge joining two vertices in A, we get a matching of size m. Therefore each vertex in A is adjacent to at least m of the vertices u_i, v_i for $1 \leq i \leq m-1$. Let x, y be any two vertices in A. Then there must exist an edge $u_i v_i$ such that there are at least 3 edges joining $\{u_i, v_i\}$ to $\{x, y\}$. This implies we can replace the edge $u_i v_i$ by either xu_i, yv_i or xv_i, yu_i to get a matching of size m in G.

   Suppose G is a graph with n vertices and more than $f(m, n)$ edges. Then $n \geq 2m$ (Kruskal-Katona). If every vertex has degree at least m, the result follows by the previous argument. On the other hand, if there exists a vertex of degree at most $m-1$, delete the endpoints of any edge incident with it. This reduces the total number of edges by at most $m - 1 + n - 2 = m + n - 3$. By simple manipulations, it can be seen that $f(m, n) - (m + n - 3) \geq f(m-1, n-2)$. Therefore by induction, the remaining graph has a matching of size $m-1$, which together with the deleted edge gives a matching of size m in G.

5. The proof is by induction on n. The statement is trivial for $n = 1$ and

it is easy to verify it for $n = 2$. Let $0 < x_1 < x_2 < \cdots < x_{n-1} < S$ be the points at which the holes are located. Note that only holes in the interval $(0, S)$ are important. Also assume without loss of generality that $a_1 < a_2 < \cdots < a_n$. The proof breaks into two cases.

Case 1. Suppose $S - a_n < x_{n-1}$. Let m be the smallest number such that $S - a_m \leq x_{n-1}$. If any of the points $S - a_i$ for $m \leq i \leq n$ is not a hole, then by induction, the grasshopper can move from 0 to $S - a_i$ using jumps $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ and avoiding the holes $x_1, \ldots, x_{n-2}$. Then a jump of a_i gives the required sequence of jumps. We may assume $S - a_i$ is a hole for $m \leq i \leq n$. In particular $S - a_n = x_j$ for some $1 \leq j \leq n - 2$. Consider the set of points $S - a_n - a_i$ for $1 \leq i < m$. If any one of these is not a hole, the grasshopper can move from 0 to $S - a_n - a_i$ using jumps $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}$ and avoiding the holes $x_1, x_2, \ldots, x_{n-3}$ then jump to $S - a_i$ and then to S. Note that $S - a_n - a_i < x_{n-2}$ but $S - a_i > x_{n-1}$ in this case. Since there are only $n - 1$ holes, one of these two cases must hold as the holes obtained for distinct values of i must be distinct.

Case 2. $S - a_n \geq x_{n-1}$. By induction, the grasshopper can move from 0 to $S - a_n$ using jumps $a_1, \ldots, a_{n-1}$ and avoiding the holes $x_1, x_2, \ldots, x_{n-2}$. If in this path, the grasshopper lands on the hole at x_{n-1}, replace the jump that takes it to x_{n-1} by a_n and then take the remaining jumps arbitrarily. Since a_n is the largest jump, replacing the jump by a_n will take the grasshopper beyond x_{n-1}. Since x_{n-1} is the furthest hole, remaining jumps can be taken arbitrarily.