Basics of Power Systems: Review of Phasors

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1. Sinusoidal Nature of Voltage and Currents
2. Back to Phasors
3. Phasor Computation
4. Behaviour of Circuit Elements
5. Working with Complex Exponential
6. Examples
Flux Linking and EMF Generation

- Flux linked to coil is given by
  \[ \phi = \int \vec{B} \cdot d\vec{s} = BA \cos \theta \]

- Suppose coil rotates around its axis at speed \( \omega \)
  \[ \theta(t) = \omega t + \theta_0 \]
  \[ \Rightarrow \phi(t) = BA \cos (\omega t + \phi) \]

- As per Faraday’s law
  \[ e = -N \frac{d\phi}{dt} \quad \text{[generator convention]} \]
  \[ = +NBA\omega \sin(\omega t + \theta_0) \]
Flux Linking and EMF Generation

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Convention

Generator

\[ e - v = 0 \]

\[ \Rightarrow v = e = -N \frac{d\phi}{dt} \]

Motor

\[ v + e = 0 \]

\[ \Rightarrow v = -e = N \frac{d\phi}{dt} = \frac{d\lambda}{dt} \]
Example of Motor Convention

\[ v = \frac{d\lambda}{dt} = L \frac{di}{dt} \]  \[ \therefore \lambda = Li \]
Voltage generated by generator at its terminal is sinusoidal in nature.

Whether it is sine or cosine depends upon where origin or $t = 0$ is placed on the sinusoid.

If there are 3 coils displaced in space by $120^\circ$, we would get 3-phase voltage,

$$v_a(t) = V_m \sin(\omega t + \phi)$$

$$v_b(t) = V_m \sin \left( \omega t + \phi - \frac{2\pi}{3} \right)$$

$$v_c(t) = V_m \sin \left( \omega t + \phi - \frac{4\pi}{3} \right)$$

$$= V_m \sin \left( \omega t + \phi + \frac{2\pi}{3} \right)$$
AC generators produces sinusoidal voltages.

The generated voltages in multi-phase system e.g. 3-phase, 5-phase, etc. have a symmetry about them.
Addition of Sinusoids at Same Frequency

Let,

\[ v_1(t) = V_{m1} \sin(\omega t + \phi_1) \]
\[ = V_{m1} \sin \omega t \cos \phi_1 + V_{m1} \cos \omega t \sin \phi_1 \]
\[ v_2(t) = V_{m2} \sin(\omega t + \phi_2) \]
\[ = V_{m2} \sin \omega t \cos \phi_2 + V_{m2} \cos \omega t \sin \phi_2 \]

Then,

\[ v_1(t) + v_2(t) = [V_{m1} \cos \phi_1 + V_{m2} \cos \phi_2] \times \sin \omega t \]
\[ + [V_{m1} \sin \phi_1 + V_{m2} \sin \phi_2] \times \cos \omega t \]
\[ = A \sin \omega t + B \cos \omega t \]
\[ = \sqrt{A^2 + B^2} \left[ \frac{A}{\sqrt{A^2 + B^2}} \sin \omega t + \frac{B}{\sqrt{A^2 + B^2}} \cos \omega t \right] \]
\[ = \sqrt{A^2 + B^2} \left[ \cos \theta \sin \omega t + \sin \theta \cos \omega t \right] \]
\[ = V_m \sin(\omega t + \theta) \]

\[ \tan \theta = \frac{V_{m1} \sin \phi_1 + V_{m2} \sin \phi_2}{V_{m1} \cos \phi_1 + V_{m2} \cos \phi_2} \]
\[ V_m^2 = (V_{m1} \cos \phi_1 + V_{m2} \cos \phi_2)^2 + (V_{m2} \sin \phi_1 + V_{m2} \sin \phi_2)^2 \]
\[ = V_{m1}^2 + V_{m2}^2 + 2 V_{m1} V_{m2} \cos (\phi_1 - \phi_2) \]
$V_{m_1}\sin\phi_1$

$V_{m_2}\sin\phi_2$

$V_{m_1}\cos\phi_1$

$V_{m_2}\cos\phi_2$

Can be inductively extended to addition of multiple sinusoids.
Root Mean Square Value

Let, \( x(t) = X_m \sin(\omega t + \phi) \); \( \omega = \frac{2\pi}{T} \). Then,

\[
X_{rms} = \frac{1}{T} \sqrt{\int_{t}^{t+T} x(\tau)^2 d\tau}
\]
What is a Phasor?

- Sinusoids at a given frequency \( x(t) = X_m \sin(\omega t + \phi) \) can be represented by vectors.
- Has two degree of freedoms viz-á-viz, amplitude \( X_m \) or rms value \( X_{rms} = \frac{X_m}{\sqrt{2}} \) and angle \( \phi \).
- This complex number or vector representation is called a phasor.
- Can be represented in polar form as well.

- Time Domain ⇔ Phasor Domain
- Need to define one reference phasor for convenience.
Let $x(t) = 1 \times \cos(\omega t)$ be reference signal. Now, represent phasors for the following:

1. $10 \cos (\omega t + 30^\circ)$
2. $10 \cos (\omega t - 30^\circ)$
3. $\frac{5}{\sqrt{2}} \sin (\omega t + 30^\circ)$

Let, $x_1(t) = X_{m_1} \sin (\omega_1 t + \phi_1)$ and $x_2(t) = X_{m_2} \sin (\omega_2 t + \phi_2)$. If $\omega_1 \neq \omega_2$, can we represent them by phasors?
Phasor representation of KCL and KVL

\[ \sum_{i=1}^{N} I_{m_i} \cos(\omega t + \phi_i) = 0 \]

\[ \Rightarrow \sum_{i=1}^{N} \bar{I}_{m_i} = 0 \]

\[ \sum_{i=1}^{N} V_{m_i} \cos(\omega t + \phi_i) = 0 \]

\[ \Rightarrow \sum_{i=1}^{N} \bar{V}_{m_i} = 0 \]
Instantaneous power at a time $t$ is given by

$$p(t) = v(t) \times i(t)$$

With $v(t) = V_m \sin(\omega t)$ and $i(t) = I_m \sin(\omega t - \phi)$

$$p(t) = V_m I_m \sin \omega t \sin(\omega t - \phi)$$

$$= \frac{V_m I_m}{2} 2 \sin \omega t \sin(\omega t - \phi)$$

$$= \frac{V_m I_m}{\sqrt{2} \sqrt{2}} [\cos \phi - \cos(2\omega t - \phi)]$$

$$= V_{rms} I_{rms} [\cos \phi - \cos(2\omega t - \phi)]$$
Instantaneous Power in Single Phase Circuit

- Instantaneous power at a time $t$ is given by

$$p(t) = v(t) \times i(t)$$

- With $v(t) = V_m \sin (\omega t)$ and $i(t) = I_m \sin (\omega t - \phi)$

$$p(t) = V_m I_m \sin \omega t \sin(\omega t - \phi)$$

$$= \frac{V_m I_m}{2} 2 \sin \omega t \sin(\omega t - \phi)$$

$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} [\cos \phi - \cos(2\omega t - \phi)]$$

$$= V_{rms} I_{rms} [\cos \phi - \cos(2\omega t - \phi)]$$

![Graph showing instantaneous power over time](image-url)

- $P_{avg}$ represents the average power over a period $T$. The graph illustrates the variation of instantaneous power $p(t)$ with time $t$. The period $T$ is divided into two halves, each representing a quarter cycle of the waveform.
We have

\[ p(t) = V_{\text{rms}} I_{\text{rms}} [\cos \phi - \cos(2\omega t - \phi)] \]

- Pulsating with twice the frequency, i.e. \(2\omega\)
- We, therefore, calculate average power over a cycle, termed as *real* or *active* power

\[ P = \frac{1}{T_0} \int_t^{t+T_0} p(\tau) d\tau = V_{\text{rms}} I_{\text{rms}} \cos \phi \]

- Unit is watt (W)
Power Factor

- We define \( S = V_{rms} I_{rms} \) as the apparent power associated with the current.
- The ratio \( \frac{P}{S} \) is in a sense efficiency of power transfer. It is called power factor.
  - Hence, the power factor is given by \( \cos \phi \in [-1, 1] \).
- Current lags voltage, we say lagging power factor; \( \cos \phi \) is positive.
- Current leads voltage, we say it as leading power factor; \( \cos \phi \) is negative.
- For example, power factor 0.5 lagging or 0.5 leading.
Active power is a product of voltage and projection of current phasor on voltage phasor

Similarly, product of voltage phasor with quadrature component of current \((VI \sin \phi)\) can be computed

- Termed as reactive power
- Significant in power system computation
Notion of Complex Power

Consider,

\[ \vec{S} = VI^* \]
\[ = V_{rms} I_{rms} \angle 0 \]
\[ = V_{rms} I_{rms} \angle \phi \]
\[ = V_{rms} I_{rms} \cos \phi + jV_{rms} I_{rms} \sin \phi \]
\[ \Rightarrow \vec{S} = P + jQ \]
\[ |\vec{S}|^2 = P^2 + Q^2 \]

- \( \cos \phi \) and \( \sin \phi \) are dimensionless. Hence, power factor has no unit.
- \( Q = VI \sin \phi \) has a unit of VA. However, it is written as VAR. It indicates VA reactive.
- In power systems, MW and MVAR are commonly used units.
Power Triangle at Different Power Factors

- **Leading**
  - Power (P)
  - Reactive Power (Q)
  - Apparent Power (S)

- **Lagging**
  - Power (P)
  - Reactive Power (Q)
  - Apparent Power (S)

- **Unity**
  - Power (P)
  - Reactive Power (Q)
  - Apparent Power (S)
Review Questions I

- Prove that R.M.S value of a sinusoid is \( \frac{1}{\sqrt{2}} \) of its peak value.
- Let a voltage source, \( v(t) = V_m \sin(\omega t + \phi) \), deliver power to a resistive circuit (resistance R) as shown in figure below.

\[
R \\
\leftarrow i(t) \\
\uparrow \quad \downarrow \\
v(t) \\
\rightarrow
\]

- Calculate the instantaneous current and power.
- Calculate the average power.
- Draw the phasor diagram depicting voltage and current phasor.
- Draw the power triangle (\( P + jQ \)).
If current lags voltage phasor which if by 30°. Would you designate it as $I_{rms} \angle -30^\circ$ or $I_{rms} \angle +30^\circ$? Justify?

Draw circuits to correspond to be following 4 quadrants of $P - Q$ diagram as shown in figure below.

![Phasor diagram](image-url)
If, The first quadrant represents a circuit which consumes active and reactive power e.g., an inductive current.

Then, The forth quadrant represents a current which consumes active power but generates reactive power e.g., a capacitive current.

What do II$^{nd}$ and III$^{nd}$ quadrant represent?
Resistor

Let,

\[ v(t) = V_m \sin(\omega t + \phi) \]

Then,

\[ i(t) = \frac{V_m}{R} \sin(\omega t + \phi) \]

\[ p(t) = v(t) \times i(t) = \frac{V_m^2}{\sqrt{2}} \sin^2(\omega t + \phi) \]

\[ P_{av} = \frac{1}{T} \int_{t}^{t+T_0} p(\tau) d\tau = V_{rms} I_{rms} \]

\[ \cos \phi = 1 \]
Resistor Cntd. . .
Following equation relates inductor voltage and current,

\[ v(t) = L \frac{di}{dt} \]

Let, \( i(t) = I_m \sin(\omega t + \phi) \)

\[ \Rightarrow v(t) = \omega L I_m \cos(\omega t + \phi) \]

\[ \Rightarrow p(t) = V_m \sin(\omega t + \phi) \times I_m \cos(\omega t + \phi) \quad [V_m = \omega L I_m] \]

\[ = \frac{V_m I_m}{2} \sin(2\omega t + 2\phi) \]

\[ \Rightarrow P_{av} = \frac{1}{T_0} \int_{0}^{T_0} p(\tau) d\tau = 0 \]

Additionally, \( S = P + jQ = VI^* = 0 + jV_{rms}I_{rms} \)
Behaviour of Circuit Elements

Inductor cntd...
Following can be said about power associated with an inductor

- Power across an inductor is pulsating at double the frequency of voltage source
- In a quarter cycle, the inductor absorbs energy and stores it as magnetic energy (when its current is increasing)
- In the next quarter cycle, it returns the same to the source
- This behavior in steady state proceeds add infinitum
Following equation relations capacitor voltage and current

\[
\begin{align*}
i_c(t) &= C \frac{dv_c}{dt} \\
\text{Let, } v_c(t) &= V_m \sin(\omega t + \phi) \\
\Rightarrow i_c(t) &= \omega CV_m \cos(\omega t + \phi) \\
\Rightarrow p(t) &= V_m I_m \sin(\omega t + \phi) \cos(\omega t + \phi) \\
&= V_{rms} I_{rms} \sin(2\omega t + 2\phi) \\
\Rightarrow P_{av} &= \frac{1}{T_0} \int_{0}^{T_0} p(\tau) d\tau = 0
\end{align*}
\]

Additionally, \( S = P + jQ = VI^* = 0 - jV_{rms}I_{rms} \)
Behaviour of Circuit Elements

Capacitor cntd...
Behaviour of Circuit Elements

Following can be said about power associated with a capacitor

- Power across a capacitor is pulsating at double the frequency of voltage source
- In a quarter cycle, the capacitor absorbs energy and stores as electrostatic energy (when its voltage is increasing)
- In the next quarter cycle, it returns the same to the source
- This behavior in steady state proceeds ad infinitum
It is often said that inductor consumes and capacitor generates reactive power.

- Average power consumed is zero.

- Voltage and current are 90° out of phase; in case of inductor voltage leads the current while voltages lags current in capacitor.
Euler’s Identity

\( e^x \) can be represented by following infinite series

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

\[
= \sum_{r=0}^{\infty} \frac{x^r}{r!}
\]

If we permit complex exponential, then with \( x = j \theta \), we have

\[
e^{j\theta} = \sum_{r=0}^{\infty} \frac{j^r \theta^r}{r!}
\]

\[
= \sum_{r=0}^{\infty} \frac{j^{2r} \theta^{2r}}{(2r)!} + \sum_{r=0}^{\infty} \frac{j^{2r+1} \theta^{2r+1}}{(2r + 1)!}
\]

\[
= \sum_{r=0}^{\infty} \frac{(j^2)^r \theta^{2r}}{(2r)!} + \sum_{r=0}^{\infty} \frac{j (j^2)^r \theta^{2r+1}}{(2r + 1)!}
\]

\[
= \sum_{r=0}^{\infty} \frac{(-1)^r \theta^{2r}}{(2r)!} + j \sum_{r=0}^{\infty} \frac{(-1)^r \theta^{2r+1}}{(2r + 1)!} \Rightarrow e^{j\theta} = \cos \theta + j \sin \theta
\]

The above is a well known Euler’s identity which plays a very important role in analysis of steady state behavior of circuits.
Euler’s Identity cntd...

It follows that

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

Hence,

\[ \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \]
\[ \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \]

If we replace \( \theta \) with \((\omega t + \phi)\) we get,

\[ |A| \cos(\omega t + \phi) = \frac{Ae^{j\phi}e^{j\omega t} + Ae^{-j\phi}e^{-j\omega t}}{2} \]

\[ \Rightarrow |A| \cos(\omega t + \phi) = \frac{\bar{A}e^{j\omega t} + \bar{A}^*e^{-j\omega t}}{2} \]

\[ \text{\(\therefore\)} \ |A| \sin(\omega t + \phi) = \frac{Ae^{j\omega t} - \bar{A}^*e^{-j\omega t}}{2j} \]

where,

\[ \bar{A} = |A|e^{j\phi} = |A| \cos \phi + j|A| \sin \phi = A_{\text{real}} + jA_{\text{imag}} \]
Euler’s Identity cntd...

Verify following

- **Multiplication**
  \[ e^{j\theta} \times e^{j\alpha} = e^{j(\theta+\alpha)} \]

- **Differentiation**
  \[ \frac{de^{j\theta}}{d\theta} = je^{j\theta} \]
Choice of Reference Wave

- Can be either $\sin \omega t$ or $\cos \omega t$
  - Just a matter of convenience
- In steady state analysis, sinusoids recur from $-\infty$ to $\infty$ in time
  - Reference is decided by that point in time axis which is designated to $t = 0$
Geometrical representation of $e^{j\omega t}$

- $\cos \omega t$ is the projection of the rotating $e^{j\omega t}$ vector on X-axis
- $\sin \omega t$ is the project of rotating unit $e^{j\omega t}$ vector on quadrature of Y-axis
- If the initial position of the unit vector is at position $\phi$, we will generate $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$
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Geometrical representation of $e^{j\omega t}$

- $\cos \omega t$ is the projection of the rotating $e^{j\omega t}$ vector on X-axis.
- $\sin \omega t$ is the project of rotating unit $e^{j\omega t}$ vector on quadrature of Y-axis.
- If the initial position of the unit vector is at position $\phi$, we will generate $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$. 
Series R-L circuit I

Claim

Consider a simple series R-L circuit described by \( Ri + L \frac{di}{dt} = v(t) \). Let, \( v(t) = \bar{A}e^{j\omega t} \) be the forcing function. The steady state response, \( i(t) \), is given by \( \bar{B}e^{j\omega t} \).

We can verify the claim as follows:

\[
R\bar{B}e^{j\omega t} + j\omega L\bar{B}e^{j\omega t} = \bar{A}e^{j\omega t}
\]

\[
\Rightarrow (R + j\omega L)\bar{B}e^{j\omega t} = \bar{A}e^{j\omega t}
\]

Hence, \( \bar{B} = \frac{\bar{A}}{R + j\omega L} \)

\[
= \frac{|\bar{A}|e^{j\phi}}{|Z|e^{j\theta}}
\]

\[
= \frac{|A|}{|Z|}e^{j(\phi - \theta)}
\]

\( \bar{A} \) is the voltage phasor.
\( R + j\omega L \) is called the impedance of the circuit (Z).
\( \bar{B} \) is called the current phasor.
Thus, the response to input $\bar{A}e^{j\omega t}$ is given by $\frac{\bar{A}}{Z}e^{j\omega t}$ i.e.

$$i(t) = \frac{|A|}{|Z|}e^{j(\omega t + \phi) - \theta}$$

$$= \frac{|A|}{|Z|} \left[ \cos(\omega t + \phi - \theta) + j \sin(\omega t + \phi - \theta) \right]$$

Due to linear nature of the circuit, the real part of $\bar{A}$ is the response to $|\bar{A}| \cos(\omega t + \phi)$ and imaginary part of it is response to $|\bar{A}| \sin(\omega t + \phi)$.

Thus if, $v(t) = V_m \cos(\omega t + \phi)$

Then, $i(t) = \frac{V_m}{|Z|} \cos(\omega t + \phi - \theta)$

and if, $v(t) = V_m \sin(\omega t + \phi)$

Then, $i(t) = \frac{V_m}{|Z|} \sin(\omega t + \phi - \theta)$

Voltage and current phasors satisfies Ohm’s law (with complex quantities).

Notice that unit of impedance is ohms.
Series R-L-C Circuit

Claim

For a series R-L-C circuit as shown in figure described by

\[ Ri + L \frac{di}{dt} + \frac{1}{C} \int_{-\infty}^{T} i(\tau)d\tau = v(t) \]

the phasor response is given by

\[ [R + j\omega L - \frac{1}{j\omega C}] \bar{I} = \bar{V} \]

Hence, complex impedance is given by

\[ Z = R + j(\omega L - \frac{1}{\omega C}) \]

What happens when \( \omega L = \frac{1}{\omega C} \)?
Consider a parallel R-L-C network. The governing network equation is given by:

\[ i(t) = \frac{v(t)}{R} + \frac{1}{L} \int_0^t v(\tau) d\tau + C \frac{dv}{dt} \]

The phasor response is given by

\[ \tilde{I} = \left[ \frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right] \tilde{V} \]

Further, with \( \tilde{Y} \) defined such that \( \tilde{I} = \tilde{Y} \tilde{V} \), impedance of the circuit is given by

\[ \tilde{Y} = \frac{1}{Z} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{R} + j(\omega C - \frac{1}{\omega L}) \]

**Admittance**

Inverse of impedance. Its unit is called mho or siemens and is denoted either by \( \frac{1}{\Omega} \) or simply \( \Omega \).

What happens if \( \omega L = \frac{1}{\omega C} \)? [Parallel Resonance]
Thank You