Fuzzy Logic - Introduction

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- fuzzy set A
- A = {(x, μ_A(x))| x $\in$ X} where μ_A(x) is called the membership function for the fuzzy set A. *X* is referred to as the universe of discourse.
- The membership function associates each element x $\in$ X with a value in the interval [0,1].

Fuzzy sets with a discrete universe

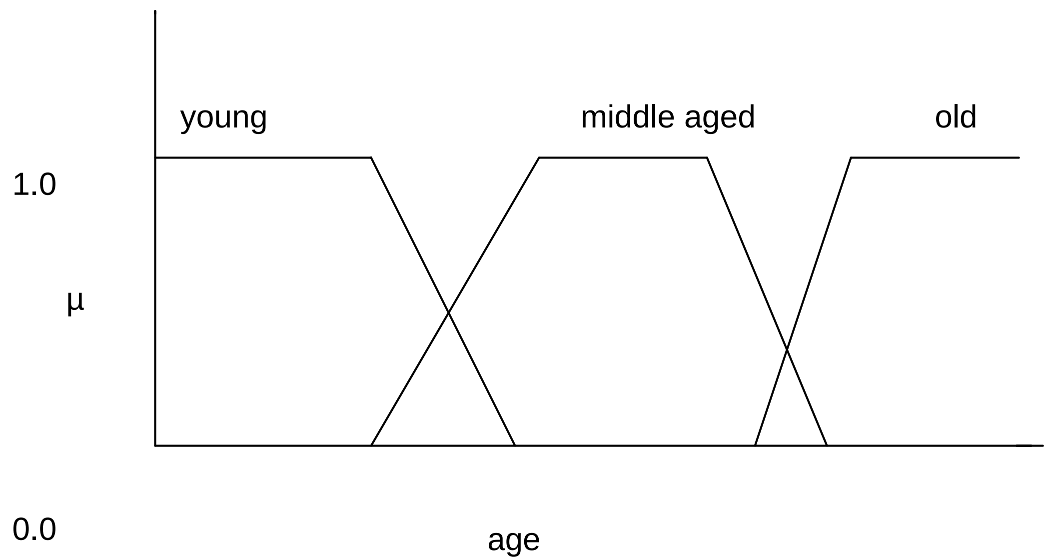
- Let X = {0, 1, 2, 3, 4, 5, 6} be a set of numbers of children a family may possibly have.
- fuzzy set A with “sensible number of children in a family” may be described by
- A = {(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)}

Fuzzy sets with a continuous universe

- $X = R+$ be the set of possible ages for human beings.
- fuzzy set B = “about 50 years old” may be expressed as
- $B = \{(x, \mu_B(x) \mid x \in X\}$, where
- $\mu_B(x) = 1/(1 + ((x-50)/10)^4$

We use the following notation to describe fuzzy sets.

- $A = \Sigma_{x_i \in X} \mu_A(x_i)/ x_i$, if X is a collection of discrete objects,
- $A = \int_X \mu_A(x)/ x$, if X is a continuous space.



young
middle aged
old
1.0
μ
0.0
age

- Support(A) is set of all points x in X such that
- $\{(x \mid \mu_A(x) > 0\}$
- core(A) is set of all points x in X such that
- $\{(x \mid \mu_A(x) = 1\}$
- Fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called fuzzy singleton

- Crossover point of a fuzzy set A is a point x in X such that
- $\{(x \mid \mu_A(x) = 0.5 \}$
- α–cut of a fuzzy set A is set of all points x in X such that
- $\{(x \mid \mu_A(x) \geq \alpha \}$

- Convexity $\mu_A(\lambda x1 + (1-\lambda)x2) \geq \min(\mu_A(x1), \mu_A(x2))$. Then A is convex.
- Bandth width is $|x2-x1|$ where x2 and x1 are crossover points.
- Symmetry $\mu_A(c+x) = \mu_A(c-x)$ for all $x \in X$. Then A is symmetric.

- **Containment or Subset:** Fuzzy set A is contained in fuzzy set B (or A is a subset of B) if $\mu_A(x) \leq \mu_B(x)$ for all x.
- **Fuzzy Intersection:** The intersection of two fuzzy sets A and B , $A \cap B$ or *A AND B* is fuzzy set C whose membership function is specified by the
- $\mu_C(x) = \mu_{A \cap B} = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$.

- **Fuzzy Union :** The union of two fuzzy sets A and B written as $A \cup B$ or *A OR B* is fuzzy set C whose membership function is specified by the
- $\mu_C(x) = \mu_{A \cup B} = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$.

- **Fuzzy Complement:** The complement of A denoted by Ā or NOT A and is defined by the membership function $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$.

Membership functions of one dimension

- A *triangular* membership function is specified by three parameters {a, b, c}:
- Triangle(x; a, b, c) = 0 if $x \leq a$;
- = (x-a)/(b-a) if $a \leq x \leq b$;
- = (c-b)/(c-b) if $b \leq x \leq c$;
- = 0 if $c \leq x$.

A *trapezoidal* membership function is specified by four parameters {a, b, c, d} as follows:

- Trapezoid(x; a, b, c, d) = 0 if $x \le a$;
- = (x-a)/(b-a) if $a \le x \le b$;
- = 1 if $b \le x \le c$;
- = (d-x)/(d-c) 0 if $c \le x \le d$;
- = 0, if $d \le x$.

A *sigmoidal* membership function is specified by two parameters {a, c}:

- Sigmoid(x; a, c) = 1/(1 + exp[-a(x-c)]) where *a* controls slope at the crossover point *x* = *c.*
- These membership functions are some of the commonly used membership functions in the fuzzy inference systems.

Membership functions of two dimensions

- One dimensional fuzzy set can be extended to form its cylindrical extension on second dimension
- Fuzzy set A = “(x,y) is near (3,4)” is
- $\mu_A(x,y) = \exp[- ((x-3)/2)^2 -(y-4)^2]$
- $\mu_A(x,y) = \exp[- ((x-3)/2)^2] \exp -(y-4)^2]$
- =gaussian(x;3,2)gaussian(y;4,1)
- This is a composite MF since it can be decomposed into two gaussian MFs

Fuzzy intersection and Union

- $\mu_{A \cap B} = T(\mu_A(x), \mu_B(x))$ where T is T-norm operator. There are some possible T-Norm operators.
- Minimum: $\min(a,b)=a \wedge b$
- Algebraic product: ab
- Bounded product: $0 \vee (a+b-1)$

- $\mu_C(x) = \mu_{A \cup B} = S(\mu_A(x), \mu_B(x))$ where *S* is called S-norm operator.
- It is also called T-conorm
- Some of the T-conorm operators
- Maximum: S(a,b) = max(a,b)
- Algebraic sum: a+b-ab
- Bounded sum: = 1 Λ(a+b)

Linguistic variable, linguistic term

- Linguistic variable:A *linguistic variable* is a variable whose values are sentences in a natural or artificial language.
- For example, the values of the fuzzy variable *height* could be *tall, very tall, very very tall, somewhat tall, not very tall, tall but not very tall, quite tall, more or less tall.*
- *Tall is a linguistic* value or primary term
- Hedges are very, more or less so on

- If *age* is a linguistic variable then its term set is
- *T(age) = { young, not young, very young, not very young,…… middle aged, not middle aged,… old, not old, very old, more or less old, not very old,…not very young and not very old,…}.*

Concentration and dilation of linguistic values

- If A is a linguistic value then operation *concentration* is defined by $CON(A) = A^2$, and *dilation* is defined by $DIL(A) = A^{0.5}$. Using these operations we can generate linguistic hedges as shown in following examples.
- ● more or less old = DIL(old);
- ● extremely old = CON(CON(CON(old))).

Fuzzy Rules

- Fuzzy rules are useful for modeling human thinking, perception and judgment.
- A fuzzy if-then rule is of the form “If x is A then y is B” where A and B are linguistic values defined by fuzzy sets on universes of discourse *X* and *Y*, respectively.
- “x is A” is called *antecedent* and “y is B” is called *consequent*.

Examples, for such a rule are

- ● If pressure is high, then volume is small.
- ● If the road is slippery, then driving is dangerous.
- ● If the fruit is ripe, then it is soft.

Binary fuzzy relation

- A binary fuzzy relation is a fuzzy set in $X \times Y$ which maps each element in $X \times Y$ to a membership value between *0* and *1*. If X and Y are two universes of discourse, then
- $R = \{((x,y), \mu_R(x, y)) \mid (x,y) \in X \times Y\}$ is a binary fuzzy relation in $X \times Y$.
- $X \times Y$ indicates cartesian product of X and Y

- The fuzzy rule "*If x is A then y is B*" may be abbreviated as $A \rightarrow B$ and is interpreted as $A \times B$.
- A fuzzy if then rule may be defined (Mamdani) as a binary fuzzy relation R on the product space $X \times Y$.
- $R = A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x)\ T\text{-}norm\ \mu_B(y) / (x,y)$.

References

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